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Reduced Differential Transform Method for Harry Dym Equation

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***Abstract**: In this paper, the reduced differential transform method is used to Harry Dym Equation. Harry Dym Equation has special importance in partial differential equation. An approach method called the reduced differential transform method (RDTM) is presented to overcome of complex calculation of this equation. The comparison of the results obtained by RDTM and exact solution shows that RDTM is a powerful method for the solution of PDEs.*

***Keywords:** Harry Dym Equation, Nonlinear PDE, Reduced Differential Transform, Numerical Solution.*

1. INTRODUCTION

The Harry Dym Equation was first proposed by Harry Dym in 1973–1974 while trying to transfer some results about isospectral flows to the string equation. Although this equation was first found by Harry Dym, it was first time published in a 1975 paper of M.D. Kruskal [3]. The Harry Dym Equation defined as below

$$u_t = u^3 u_{xxx} \tag{1}$$

Then, we will apply the reduced differential transform method (RDTM) (see [1], [2], [6]) to solve this equation numerically.

Harry Dym Equation has been studied by a lot of researcher until now. Recently, Reza Mokhtari has constituted exact travelling wave solutions of the Harry–Dym equation with the methods of Adomian decomposition, He's variational iteration, direct integration, and power series in [5], Delara Soltani and Majid Akbarzadeh Khorshidi in [7] has shown that the approximate analytical solution of Harry Dym Equation has found by using reconstruction of variational iteration method (RVIM) and homotopy perturbation method (HPM), Kristina Mallory and Robert A. Van Gorder has obtained approximate solutions to the Dym Equation, and associated initial value problem, for general initial data by way of an optimal homotopy analysis method [4].

2. ANALYSIS OF THE METHOD

If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space *x* in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \tag{2}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.
The basic definitions of reduced differential transform method are introduced in [1], [2], [6].

Table 1: Basic transformations of Reduced Differential Transform Method.

Functional Form	**Transformed Form**
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α is a constant)
$w(x,y) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
$w(x,y) = x^m t^n u(x,t)$	$W_k(x) = x^m U(k-n)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)...(k+r)U_{k+1}(x) = \frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{\partial}{\partial x}U_k(x)$

$Nu(x,t)$	**Maple Code for Nonlinear Function** **restart;** **NF:=Nu(x,t):#Nonlinear Function** **m:=5: # Order** **u[t]:=sum(u[b]*t^b,b=0..m):** **NF[t]:=subs(Nu(x,t)=u[t],NF):** **s:=expand(NF[t],t):** **dt:=unapply(s,t):** **for i from 0 to m do** **n[i]:=((D@@i)(dt)(0)/i!):** **print(N[i],n[i]); #**Transform Function **od:**

3. APPLICATION OF RDTM

In order to assess the advantages and accuracy of RDTM, we consider the Harry Dym Equation as

$$u_t = u^3 u_{xxx} \tag{3}$$

with initial condition

$$u(x,0) = \left(a - \frac{3\sqrt{b}}{2} x \right)^{\frac{2}{3}} \tag{4}$$

For the solution procedure, we first take the differential transform of (3) by the use of Table 1 and then we have the following equation

$$(k+1)U_{k+1}(x) = N_k(x) \tag{5}$$

where $U_k(x)$ and $N_k(x)$ are the transformations of the functions $u(x,t)$ and $Nu(x,t) = u^3 u_{xxx}$ respectively. $N_k(x)$ is the nonlinear function. Let, Calculation of the nonlinear term $N_k(x)$ was given maple code in the Table 1

$$N_o = u_0^3\left(\frac{\partial^3}{\partial x^3}u_0\right)$$

$$N_1 = 3u_0^2u_1\left(\frac{\partial^3}{\partial x^3}u_0\right)+u_0^3\left(\frac{\partial^3}{\partial x^3}u_1\right)$$

$$N_2 = \left(3u_0^2u_2+3u_0u_1^2\right)\left(\frac{\partial^3}{\partial x^3}u_0\right)+3u_0u_1^2\left(\frac{\partial^3}{\partial x^3}u_1\right)+u_0^3\left(\frac{\partial^3}{\partial x^3}u_2\right)$$

$$\vdots$$

and the exact solution of problem is $u(x,t)=\left(a-\frac{3\sqrt{b}}{2}(x+bt)\right)^{\frac{2}{3}}$.

From the initial and boundary condition (4), we write

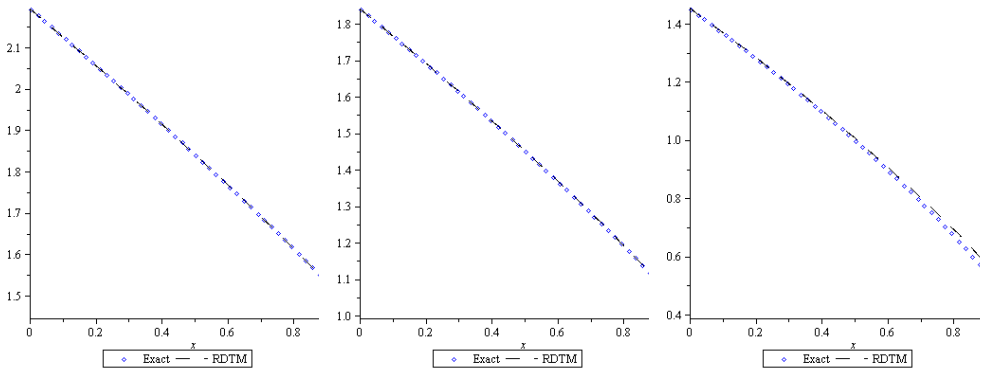
$$U_0(x)=\left(a-\frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}} \tag{6}$$

Substituting (6) into (5), we obtain the following $U_k(x)$ values successively. Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{5}$ gives five term (Order 5) approximation solution as

$$\tilde{u}_5(x,t)=-\frac{1}{3}\left(\begin{array}{l}162b^4t^2ax^2-576b^2ta^3x+486b^{\left(\frac{7}{2}\right)}tx^4-81b^{\left(\frac{9}{2}\right)}t^2x^3+36b^{\left(\frac{11}{2}\right)}t^3x^2\\+3240a^2b^{\left(\frac{3}{2}\right)}x^3-2430ab^2x^4+24b^3t^2a^3+14b^6t^4a-2160a^3bx^2\\-1296b^3tax^3-48b^5t^3ax+96b^{\left(\frac{3}{2}\right)}ta^4+16b^{\left(\frac{9}{2}\right)}t^3a^2-21b^{\left(\frac{13}{2}\right)}t^4x-96a^5\\+720a^4b^{\left(\frac{1}{2}\right)}x+729b^{\left(\frac{5}{2}\right)}x^5-108b^{\left(\frac{7}{2}\right)}t^2a^2x+1296b^{\left(\frac{5}{2}\right)}ta^2x^2+14b^{\left(\frac{15}{2}\right)}t^5\end{array}\right)$$
$$/\left((-2a+3b^{\left(\frac{1}{2}\right)}x)^4(8a-12b^{\left(\frac{1}{2}\right)}x)^{\frac{1}{3}}\right)$$

Therefore, the exact solution of problem is given by

$$u(x, y) = \lim_{n \to \infty} \tilde{u}_n(x, y) .$$



$t = 0.5$ $t = 1.0$ $t = 1.5$

Figure 1: The comparison of the RDTM and Exact Solution various t values.

4. CONCLUSIONS

In this paper, the reduced differential transform method has been successfully used to obtain the approximate analytical and exact solutions of Harry Dym equation. The result obtained here was compared with the exact solution. The result showed that RDTM is powerful mathematical tool for solutions of nonlinear partial differential equations in terms of preciseness and sufficiency. The main advantage of the method is providing to its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with smoothly computed terms. In our work, we made use of the Maple Package to calculate the series obtained from the reduced differential transform method.

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New Approach for Numerical Solution of non-Newtonian Nanofluid Flow Between Two Vertical Plates

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***Abstract**: In this study, variational iteration and modified variational iteration methods are implemented to solve nonlinear differential system arising from the sodium alginate based non-newtonian nanofluid flow between two vertical plates. By making use of literature at first, we will describe the problem and second reduce the problem to nonlinear ordinary differential system with similarity variables transform. And then, we will solve the system by using variational iteration and modified variational iteration methods. Also, our results obtained from variational iteration and modified variational iteration methods are compared with well-known numerical method in terms of the accuracy and effectiveness.*

***Keywords:** Nonlinear differential equation, Variational Iteration Method, Numerical solution, Non-Newtonian Fluid Flow, Modified Variational Iteration Method.*

1. INTRODUCTION

Computational Fluid Dynamics (CFD) is a very important branch of fluid mechanics which uses numerical methods and algorithms to solve and analyze problems that include fluid flows. CFD can be used various areas such as boundary layer flow, Newtonian-nonnewtonian nanofluid flow, heat transfer and viscous flow, magnetohydrodynamic fluid flow, laminar boundary layer flow and etc.

In literature, Hatami and Ganji (2014), Hatami and Ganji (2013), Hatami et al. (2013), Ziabakhsh and Ganji (2009), Xu and Liao (2005), Yoshino et al. (2007) performed numerical methods such as fourth order Runge-Kutta, Runge-Kutta-Fehlberg, optimal homotopy analysis, differential transform-Pade, Least Square methods etc. with regard to Computational Fluid Dynamics (see [1-6]).

In present study, the system of nonlinear differential equations come from natural non-newtonian nanofluid flow between two vertical plates is considered. Recently, this problem has been studied by Hatami and Ganji [1] with Differential Transform and Least Square methods. By applying similarity transforms, we reduce this system of equations into set of nonlinear ordinary differential equations. To solve these equations numerically, VIM and MVIM are implemented. Very accurate and convergent results are obtained. Also, obtained numerical results are compared with well-known methods.

2. DESCRIPTION OF PROBLEM

The problem occurs two vertical plates separated by a $2b$ distance apart. The walls at $x=+b$ and $x=-b$ are held at constant temperatures T_2 and T_1 respectively, here $T_1 > T_2$. The fluid is a non-Newtonian Sodium Alginate (SA) based nanofluid containing Cu and Ag nanoparticles.

The effective density ρ_{nf} , the effective dynamic viscosity μ_{nf} , the heat capacitance $(\rho \mathrm{C}_p)_{nf}$ and thermal conductivity k_{nf} of the nanofluid are described in [1,2,3]. By considering [1,8], we define the similarity variables

$$G=\frac{\upsilon}{G_0},\quad \eta=\frac{x}{b},\quad \theta=\frac{T-T_m}{T_1-T_2}. \tag{1}$$

Taking in the account of these assumptions and further by considering Maxwell-Garnetts model [9], [1], Navier-Stokes and energy equations can be reduced as

$$\begin{aligned}
&\frac{d^2G}{d\eta^2}+6\varepsilon\left(1-\phi\right)^{\frac{5}{2}}\left(\frac{dG}{d\eta}\right)^2\frac{d^2G}{d\eta^2}+\theta=0\\
&\frac{d^2\theta}{d\eta^2}+Ec\,\mathrm{Pr}\left(\frac{\left(1-\phi\right)^{-\frac{5}{2}}}{H}\right)\left(\frac{dG}{d\eta}\right)^2+\frac{2\varepsilon Ec\,\mathrm{Pr}}{H}\left(\frac{dG}{d\eta}\right)^4=0
\end{aligned} \tag{2}$$

with boundary conditions

$$\begin{aligned}
&\eta=-1:\ \ G=0,\ \ \theta=0.5\\
&\eta=1:\ \ G=0,\ \ \theta=-0.5
\end{aligned} \tag{3}$$

where ϕ is solid volume fraction, (Pr) Prandtl number, (Ec) Eckert number, ε non-Newtonian viscosity.

3. MATHEMATICAL METHODS

3.1. Variational Iteration Method

Variational iteration method (VIM) is one of the powerful mathematical tool to solve various kinds of linear and nonlinear problems which was proposed, for the first time, by He [11-13].

In order to basic definition of VIM, we consider the following general nonlinear problem [11-13]

$$L[u(x)]+N[u(x)]=g(x) \tag{4}$$

where L is a linear operator, N is a nonlinear operator and g is a given continuous function. According to the originally VIM, we construct correction functional as [11-13]

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(s)\left(L(u_n(s))+N(\tilde{u}_n(s))-g(s)\right)ds\ ,\ n\geq 0 \tag{5}$$

Here, λ is a Lagrange multiplier [11], $\tilde{u}_n$ is considered as a restricted variation [2], i.e. $\delta\tilde{u}_n=0$. Firstly, determine the Lagrange multiplier which can be identified optimally via variational theory [11]. Then, successive iterations $u_n(x),\ n\geq 0$ are obtained by using Lagrange multiplier and initial approximate function u_0 that satisfies, at least, the initial and boundary conditions. Consequently, the exact solution of (4) can be obtained by using [11-13]

$$u(x)=\lim_{n\to\infty} u_n(x)\ . \tag{6}$$

3.2. Modified Variational Iteration Method

According to VIM procedures, determination of optimal Lagrange multiplier, restricted variations and initial approximation are very important for

convergence and accuracy of method. Therefore, we rearrange the equation (4) to identify new Lagrange multiplier for efficient modification of VIM as [10]

$$L[u(x)]+L_1[u(x)]-L_1[u(x)]+N[u(x)]=g(x) \tag{7}$$

where $L_1[u(x)]$ is arbitrary linear operator of $u(x)$. Then, we can construct the new correction functional as [10]

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(s)\begin{pmatrix} L(u_n(s))+L_1(u_n(s)) \\ -L_1(\tilde{u}_n(s))+N(\tilde{u}_n(s))-g(s) \end{pmatrix} ds \ , n\geq 0 \tag{8}$$

Here, the Lagrange multiplier obtained from the correction functional (8) is different from that obtained from (5). In this way, it may be shown that the auxiliary linear operator can choose arbitrarily.

4. SOLUTION OF PROBLEM BY VIM AND MVIM

Let, we apply the VIM and MVIM procedure (4)-(8) to eqs. (2)-(3). Then, the initial approximation $G_0(\eta),\ \theta_0(\eta)$, Lagrange multiplier $\lambda_G(s),\ \lambda_\theta(s)$ and correct functional are obtained as respectively

$$G_0(\eta)=a+b\eta,\ \theta_0(\eta)=c+d\eta \tag{9}$$

$$\begin{aligned} &\text{VIM}\rightarrow \lambda_G(s)=s-\eta,\ \lambda_\theta(s)=s-\eta \\ &\text{MVIM}\rightarrow \lambda_G(s)=\sin(s-\eta),\ \lambda_\theta(s)=\sin(s-\eta) \end{aligned} \tag{10}$$

$$G_{n+1}(\eta)=G_n(\eta)+\int_0^\eta \lambda(s)\begin{pmatrix} G_n''(s)+6\varepsilon(1-\phi)^{\frac{5}{2}} \\ (G_n'(s))^2 G_n''(s)+\theta_n(s) \end{pmatrix} ds \ , n\geq 0$$

$$\theta_{n+1}(\eta)=\theta_n(\eta)+\int_0^\eta \lambda(s)\begin{pmatrix} \theta_n''(s)+Ec\,\mathrm{Pr}\left(\dfrac{(1-\phi)^{-\frac{5}{2}}}{H}\right)(G_n'(s))^2 \\ +\dfrac{2\varepsilon Ec\,\mathrm{Pr}}{H}(G_n'(s))^4 \end{pmatrix} ds \tag{11}$$

where prime denotes the derivation with respect to s, a,b,c,d are unknown coefficients which can be found by applying boundary conditions (3). By using

(9)-(10) into (11) and from iteration of (11), we obtain first iterations of VIM and MVIM respectively

$$\text{VIM} \to G_1(\eta) = a + b\eta - \frac{(3c\eta^2 + d\eta^3)}{6},$$

$$\text{VIM} \to \theta_1(\eta) = c + d\eta + \left(-\frac{1}{2}\frac{Ec\,\text{Pr}\,b^2}{H\left(1-\phi\right)^{2.5}} - \frac{\varepsilon Ec\,\text{Pr}\,b^4}{H} \right)\eta^2$$

(13) $$\text{MVIM} \to G_1(\eta) = a - c + \left(b - d\right)\eta + c\cos\left(\eta\right) + d\sin\left(\eta\right)$$

$$\text{MVIM} \to \theta_1(\eta) = c + d\eta + \left(\frac{Ec\,\text{Pr}\,b^2\left[1 + 2\varepsilon b^2\left(1-\phi\right)^{2.5}\right]\left(\cos\left(\eta\right)-1\right)}{H\left(1-\phi\right)^{2.5}} \right)$$

This iteration process continues sufficiently (as seen (6)) to obtain solution of (2-3).

Table 1: Comparison of $f(\eta)$ (velocity profile) values.

η	Numerical[1][1]	DTM[1]	LSM[1]	Present(VIM)	Present(MVIM)
–0.9	0.013888702	0.01402428	0.0137141	0.01459283555	0.0132452718
–0.7	0.029489465	0.02945437	0.0290255	0.03067024278	0.0282671690
–0.5	0.031279886	0.03127150	0.0307996	0.03260329822	0.0303561886
–0.3	0.023053394	0.02332968	0.0225728	0.02439200185	0.0227837273
–0.1	0.008866496	0.00948289	0.0082130	0.01003635367	0.00920635436
0.0	0.000776279	0.00154951	0.0000001	0.00180439764	0.00137817278
0.1	-0.007333249	-0.00641485	-0.0082121	-0.00646364633	-0.00647996380
0.3	-0.021672314	-0.02050956	-0.0225753	-0.02110799815	-0.0202951841
0.5	-0.030190188	-0.02894723	-0.0307938	-0.02989670176	-0.0283338576
0.7	-0.028801308	-0.02787389	-0.0290255	-0.02882975720	-0.0269208288
0.9	-0.013660270	-0.01343546	-0.0137142	-0.01390716445	-0.0127577514

[1]Fourth order Runge-Kutta

5. CONCLUSIONS

We applied the variational iteration (VIM) and modified variational (MVIM) methods for nonlinear differential equation system related to non-Newtonian nanofluid flow between two vertical plates to solve approximately. We compared our results with differential transform method (DTM), least square

method (LSM) and fourth order Runge-Kutta methods. Numerical results obtained from VIM and MVIM have higher accuracy and efficient convergence with compared fourth order Runge-Kutta which is a well-known exact solution method as shown Table 1 and Table 2. All of the calculation has been made only one step iteration by VIM and MVIM. Therefore, it's clearly seen that VIM and MVIM are very effective and powerful tool.

Table 2: Comparison of $\theta(\eta)$ (temperature profile) values.

η	Numerical[1][1]	DTM[1]	LSM[1]	Present(VIM)	Present(MVIM)
–0.9	0.45043148	0.45059686	0.45	0.4506856711	0.4505302619
–0.7	0.35094362	0.35161056	0.35	0.3518404856	0.3514643756
–0.5	0.25135346	0.25238090	0.25	0.2527065965	0.2521996315
–0.3	0.15173412	0.15290389	0.15	0.1532840037	0.1527067172
–0.1	0.05197647	0.05317557	0.05	0.05357270734	0.05296541684
0.0	0.002010267	0.00321593	0.0	0.003608795288	0.002997998036
0.1	-0.04802206	-0.04680801	-0.05	-0.04642729266	-0.04703458316
0.3	-0.14826378	-0.14705093	-0.15	-0.1467159963	-0.1472932828
0.5	-0.24864875	-0.24755700	-0.25	-0.2472934035	-0.2478003685
0.7	-0.34906512	-0.34833031	-0.35	-0.3481595144	-0.3485356244
0.9	-0.44957637	-0.44937485	-0.45	-0.4493143289	-0.4494697381

[1]Fourth order Runge-Kutta

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Some Properties Concerning $(3,2,\rho)$-K-metrizable Spaces

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***Abstract**: We define a locally $(3,2,\rho)$-K-metrizable space. We prove that a locally $(3,2,\rho)$-K-metrizable space has a σ-locally finite basis. As a corollary we obtain that a T_1-$(3,2,\rho)$-K-metrizable space is metrizable. We prove that the Lindelöf property, second-countability and separability are equivalent in $(3,2,\rho)$-K-metrizable spaces.*

***Keywords:** $(3,2,\rho)$-K-metric, locally $(3,2,\rho)$-K-metrizable space*

1. INTRODUCTION

The geometric properties, their axiomatic classification and the generalization of metric spaces have been considered by: K. Menger ([14]), V. Nemytzki, P.S. Aleksandrov ([15], [1]), Z. Mamuzic ([13]), S. Gäler ([11]), A.V. Arhangelskii, M. Choban, S. Nedev ([2], [5], [17]), B.C. Dhage, Z. Mustafa, B. Sims ([6], [15]), and many others. The notion of (n,m,ρ)-metric is introduced in [7]. Connections between some of the topologies induced by a $(3,1,\rho)$-metric and topologies induced by a pseudo-o-metric, o-metric and symmetric are given in [8]. For a given $(3,j,\rho)$-metric d on a set M, $j\in\{1,2\}$, seven topologies $\tau(G,d),\tau(H,d),\tau(D,d),\tau(N,d)$, $\tau(W,d)$, $\tau(S,d)$ and $\tau(K,d)$ on M, induced by d, are defined in [4], and several properties of these topologies are shown.

In this paper we are concerned with $(3,2,\rho)$-K-metrizable spaces (M,τ), i.e. with topological spaces (M,τ) such that $\tau=\tau(K,d)$ for a $(3,2,\rho)$-metrics d on M. We define a locally $(3,2,\rho)$-K-metrizable space and prove the following results:

i) Any locally $(3,2,\rho)$-K-metrizable space has a σ-locally finite basis;

ii) Any T_1-$(3,2,\rho)$-K-metrizable space is metrizable;

iii) A space is $(3,2)$-K-metrizable iff it is metrizable; and

iv) In any $(3,2,\rho)$-K-metrizable space (M,τ) the following conditions are equivalent: **a)** (M,τ) has the Lindelöf property; **b)** (M,τ) is second-countable; **c)** (M,τ) is separable.

2. $(3,2,\rho)$-K-METRIZABLE SPACES

In this paper $\mathbf{R}_0^+$ denotes the set of all nonnegative real numbers, $\mathbf{Q}$ the set of all rational numbers and $\mathbf{N}$ the set of all natural numbers.

Let $M \neq \emptyset$. We denote by $M^{(3)}$ the symmetric third power of M, i.e. $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$(x,y,z)\alpha(u,v,w) \Leftrightarrow (u,v,w)$ is a permutation of (x,y,z).

Let $d: M^{(3)} \to \mathbf{R}_0^+$. We state two conditions for such map:

(M0) $d(x,x,x) = 0$, for any $x \in M$;

(M1) $d(x,y,z) \leq d(x,a,b) + d(a,y,b) + d(a,b,z)$, for any $x,y,z,a,b \in M$.

For a map d as above let $\rho = \{(x,y,z) \mid (x,y,z) \in M^{(3)}, d(x,y,z) = 0\}$. The set ρ is a (3,2)-equivalence on M defined as in [7], and discussed in [8]. The set $\Delta = \{(x,x,x) \mid x \in M\}$ is a $(3,2)$-equivalence on M.

Definition 2.1. If $d: M^{(3)} \to \mathbf{R}_0^+$ and ρ are as above and d satisfies (M0) and (M1) we say that d is a $(3,2,\rho)$**-metric** on M. If d is a $(3,2,\Delta)$-metric on M, we say that d is $(3,2)$**-metric** on M.

Example 2.1. Let $M \neq \emptyset$. The map $d: M^{(3)} \to \mathbf{R}_0^+$ defined by:

$d(x,y,z) \leq \begin{cases} 0, (x,y,z) \in \Delta \\ 1, (x,y,z) \notin \Delta \end{cases}$ is a $(3,2)$-metric on M (**discrete 3-metric**).

It is easy to check that if d is a $(3,2,\rho)$-metric on M, then for any $x,y,z \in M$, $d(x,x,y) \leq 2d(x,y,y)$ and $d(x,x,y) \leq 2d(x,z,z) + d(y,z,z)$.

Let d be a $(3,2,\rho)$-metric on M, $x \in M$ and $\varepsilon > 0$. We define, as in [4], a "little" ε-ball with centre at x and radius ε by $L(x,\varepsilon) = \{y \mid y \in M, d(x,y,y) < \varepsilon\}$.

Next, we define two topologies on M induced by d as follows:

1) $\tau(K,d)$-the topology on M generated by all the ε-balls $L(x,\varepsilon)$;

2) $\tau(S,d)$-the topology defined by:

$U \in \tau(S,d) \Leftrightarrow (\forall x \in U)(\exists \varepsilon > 0)\ L(x,\varepsilon) \subseteq U$.

Proposition 2.1. The set $\{L(x,\varepsilon) \mid x \in M, \varepsilon > 0\}$ is a base for $\tau(K,d)$, moreover $\tau(K,d) = \tau(S,d)$.

Proof. It is enough to show that $L(x,\varepsilon)\in\tau(S,d)$ for any $x\in M,\varepsilon>0$. Let $y\in L(x,\varepsilon)$ and let $4\delta=\varepsilon-d(x,y,y)$. Then, for each $z\in L(y,\delta)$, $d(x,z,z)\le d(x,y,y)+2d(z,y,y)\le d(x,y,y)+4d(y,z,z)<d(x,y,y)+4\delta$ and $d(x,y,y)<\varepsilon-4\delta$ implies that $d(x,z,z)<\varepsilon$, i.e. $z\in L(y,\delta)\subseteq L(x,\varepsilon)$. All this shows that $L(x,\varepsilon)\in\tau(S,d)$.

Definition 2.2. We say that the topological space (M,τ) is $(3,2,\rho)$-**K-metrizable** if there is $(3,2,\rho)$-metric d such that $\tau=\tau(K,d)$.

3. SOME PROPERTIES CONCERNING $(3,2,\rho)$-K-METRIZABLE SPACES

Definition 3.1. We say that the topological space (M,τ) is **locally** $(3,2,\rho)$ **-K-metrizable** if every point $x\in M$ has a neighborhood $U\in\tau$ such that the subspace (U,τ_U) is $(3,2,\rho)$-K-metrizable, where $\tau_U=\{V\cap U\mid V\in\tau\}$.

Proposition 3.1. If (M,τ) is paracompact locally $(3,2,\rho)$-K-metrizable space, then (M,τ) has a σ-locally finite basis.

Proof. Let $\boldsymbol{U}$ be an open covering of (M,τ) consisting of $(3,2,\rho)$-K-metrizable subsets of M. Then there exist a locally finite open refinement $\boldsymbol{C}$ of $\boldsymbol{U}$ that covers M (paracompactness of M).

Let $C\in\boldsymbol{C}$ be a $(3,2,\rho)$-K-metrizable subspace and $d_C:C\times C\times C\to\mathbf{R}_0^+$ be a $(3,2,\rho)$-metric that gives the topology on C. It's clear that the ball $L_C(x,\varepsilon)=\{y\in C\mid d_C(x,y,y)<\varepsilon\}$ is an open set in τ_C, because $\tau_C=\tau_C(K,d_C)=\tau_C(S,d_C)$.

For each $m\in\mathbf{N}$ let $\boldsymbol{A}_m=\{L_C(x,1/m)\mid x\in C,C\in\boldsymbol{C}\}$. Then $\boldsymbol{A}_m$ is an open covering of M. There exist locally finite open refinement $\boldsymbol{D}_m$ of $\boldsymbol{A}_m$ that covers M (again paracompactness of M). Let $\boldsymbol{D}=\bigcup_{n=1}^{\infty}\boldsymbol{D}_m$. Then $\boldsymbol{D}$ is a σ-locally finite family.

Next we will show that $\boldsymbol{D}$ is a basis for τ. Let $x\in M$ and let U be a neighborhood of x. Since $\boldsymbol{C}$ is locally finite, x belongs to only finetly many elements $C_1,C_2,...,C_k$ of $\boldsymbol{C}$. Then $x\in U\cap C_i$ and $U\cap C_i$ is an open set in C_i. So, there exists $\varepsilon_i>0$ such that $L_{C_i}(x,\varepsilon_i)\subseteq U\cap C_i$. We choose $m\in\mathbf{N}$

such that $6/m < \min\{\varepsilon_i \mid i = \overline{1,k}\}$. Since $\boldsymbol{D}_m$ covers M, there exists $D \in \boldsymbol{D}_m$ such that $x \in D$. Because $\boldsymbol{D}_m$ refines $\boldsymbol{A}_m$, there are element $C \in \boldsymbol{C}$ and some $y \in C$ such that $x \in D \subseteq L_C(y,1/m)$. From the definition of $L_C(y,1/m)$ it follows that $x \in C$. So, $C = C_i$ for some $i \in \{1,2,...,k\}$. For $z \in L_{C_i}(y,1/m)$, we have that $d_{C_i}(y,z,z) < 1/m$ and $d_{C_i}(z,y,y) \le 2d_{C_i}(y,z,z) < 2/m$. Since $x \in L_{C_i}(y,1/m)$, we have that $d_{C_i}(x,y,y) \le 2d_{C_i}(y,x,x) < 2/m$. All this imply that $d_{C_i}(x,z,z) \le d_{C_i}(x,y,y) + 2d_{C_i}(z,y,y) \le 6/m < \varepsilon_i$, i.e. $z \in L_{C_i}(x,\varepsilon_i)$. So, we have shown that $x \in D \subseteq L_{C_i}(y,1/m) \subseteq L_{C_i}(x,\varepsilon_i) \subseteq U \cap C_i \subseteq U$, i.e. $\boldsymbol{D}$ is a basis for τ.

Corollary 3.1. A T_1-$(3,2,\rho)$-K-metrizable space is metrizable.

Proof. If (M,τ) is T_1-$(3,2,\rho)$-K-metrizable, then it is T_2, parcompact ([9]) and locally $(3,2,\rho)$-K-metrizable space. So, it is T_3-space with σ-locally finite basis. From Nagata-Smirnov metrization theorem and previous proposition it follows that (M,τ) is metrizable.

Corollary 3.2. A space is $(3,2)$-K-metrizable iff it is metrizable.

Proposition 3.2. A $(3,2,\rho)$-K-metrizable space (M,τ) has the Lindelöf property iff it is second-countable.

Proof. Let (M,τ) be a Lindelöf space. For each $n \in \mathbf{N}$, the collection $\boldsymbol{B}_n = \{L(x,1/n) \mid x \in M\}$ is an open covering for M. Then there exists $\boldsymbol{B}_n^* = \{L(x_m^n,1/n) \mid m \in \mathbf{N}\}$ a countable subcollection of $\boldsymbol{B}_n$ that covers M.

We will prove that $\boldsymbol{B}^* = \bigcup_{n=1}^{\infty} \boldsymbol{B}_n^*$ is a basis for (M,τ), i.e. for each open ball $L(x,\varepsilon)$ there is $O \in \boldsymbol{B}^*$ such that $x \in O \subseteq L(x,\varepsilon)$. For $x \in M$ and $\varepsilon > 0$ there exists $L(x_{m_0}^{n_0},1/n_0)$ such that $x \in L(x_{m_0}^{n_0},1/n_0)$ and $6/n_0 < \varepsilon$. Let $O = L(x_{m_0}^{n_0},1/n_0)$ and $y \in O$. Then we have that $d(y,y,x_{m_0}^{n_0}) < 1/n_0$ and $d(y,y,x) \le 2d(y,x_{m_0}^{n_0},x_{m_0}^{n_0}) + d(x,x_{m_0}^{n_0},x_{m_0}^{n_0}) \le 4d(y,y,x_{m_0}^{n_0}) + 2d(x,x,x_{m_0}^{n_0}) < 4/n_0 + 2/n_0 = 6/n_0 < \varepsilon$. Hence, $y \in L(x,\varepsilon)$, i.e. $x \in O \subseteq L(x,\varepsilon)$.

The opposite direction is clearly true in any topological space.

Proposition 3.3. A $(3,2,\rho)$-K-metrizable space (M,τ) is separable iff it is second-countable.

Proof. Let (M,τ) be a separable space. Then there exists a countable dense subset A in M. The collection $\boldsymbol{B}=\{L(x,\delta)\mid x\in A,\delta\in \mathbf{Q}\}$ is clearly countable.

We will prove that $\boldsymbol{B}$ is a basis for (M,τ). Let $G\subseteq M$ be an open set and $y\in G$. Then there exists $\varepsilon>0$ such that $L(y,\varepsilon)\subseteq G$. Choose $\delta_0\in \mathbf{Q}$ such that $0<5\delta_0<\varepsilon$. From $y\in M=\overline{A}$ it follows that there is $x_0\in L(y,\delta_0/2)\cap A$. Then $y\in L(x_0,\delta_0)\subseteq L(y,\varepsilon)\subseteq G$, because $d(x_0,y,y)\le 2d(x_0,x_0,y)<\delta_0$ and from $z\in L(x_0,\delta_0)$ it follows that $d(z,z,y)\le 2d(z,x_0,x_0)+d(y,x_0,x_0)\le 4d(z,z,x_0)+d(y,x_0,x_0)<5\delta_0<\varepsilon$, i.e. $z\in L(y,\varepsilon)$.

The opposite direction is clearly true in any topological space.

Corollary 3.3. In any $(3,2,\rho)$-K-metrizable space (M,τ) the following conditions are equivalent:

a) (M,τ) has the Lindelöf property;

b) (M,τ) is second-countable;

c) (M,τ) is separable.

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Separation Properties for Some Topologies Induced by (3,j,ρ)-metrics, j∈{1,2}

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***Abstract**: For a given (3,j,ρ)-metric d on a set M, j∈{1,2} we examine necessary and sufficient conditions for d providing some separation properties of the topologies on M induced by the (3,j,ρ)-metric d.*

***Keywords:** (3,j,ρ)-metric, metric space, (3,j,ρ)-metrizable spaces.*

1. INTRODUCTION

The geometric properties, their axiomatic classification and the generalization of metric spaces have been considered in a lot of papers. We will mention some of them: K. Menger ([20], [21]), V. Nemytzki, P.S. Aleksandrov ([24], [25], [1]), Z. Mamuzic ([18], [19]), S. Gähler ([13], [15]), A.V. Arhangelskii, M. Choban, S. Nedev ([2], [5], [26], [27], [28]), R. Kopperman ([17]), J. Usan ([29]), B.C. Dhage, Z. Mustafa, B. Sims ([8], [22], [23]). The notion of (n,m,ρ)-metric is introduced in [9]. Connections between some of the topologies induced by a (3,1,ρ)-metric d and topologies induced by a pseudo-o-metric, o-metric and symmetric are given in [10]. For a given (3,j,ρ)-metric d on set M, j∈{1,2}, seven topologies $\tau(G,d)$, $\tau(H,d)$, $\tau(D,d)$, $\tau(N,d)$, $\tau(W,d)$, $\tau(S,d)$ and $\tau(K,d)$ on M, induced by d, are defined in [6], and several properties of these topologies are shown.

In this paper we consider only the topologies $\tau(G,d)$, $\tau(D,d)$, $\tau(N,d)$ and $\tau(W,d)$, examine some separation properties for some of these topologies, and give necessary and sufficient condition for the (3,j,ρ)-metric such that: a) a (3,1,ρ)-N-metrizable space is T_0 (Kolmogorov) and T_1 (Freshet) space; b) a (3,1,ρ)-W-metrizable space to be T_1 space; c) a (3,1,ρ)-G-metrizable space to be T_2 (Hausdorff) space. Also we prove that: a) any (3,j)-D-metrizable space (M,τ), j∈{1,2}, is T_1 space; b) any (3,2)-D-metrizable space (M,τ) is T_2 space; and c) any (3,2)-N-metrizable space (M,τ) is perfectly normal.

2. TOPOLOGIES INDUCED BY (3,J,ρ)-METRICS

In this part we state the notions (defined in [6]) used latter.

Let M be a nonempty set, and let d: $M^3 \to R_0^+ = [0, \infty)$. We state four conditions for such a map.

(M0) d(x,x,x)=0, for any $x \in M$;

(P) d(x,y,z) = d(x,z,y) = d(y,x,z) for any $x,y,z \in M$;

(M1) $d(x,y,z) \leq d(x,y,a) + d(x,a,z) + d(a,y,z)$, for any $x,y,z,a \in M$; and

(M2) $d(x,y,z) \leq d(x,a,b) + d(a,y,b) + d(a,b,z)$, for any $x,y,z,a,b \in M$.

For a map d as above let $\rho = \{(x,y,z) \mid (x,y,z) \in M^3, d(x,y,z)=0\}$. The set ρ is a (3,j)-equivalence on M, as defined and discussed in [9], [6]. The set $\Delta=\{(x,x,x) \mid x \in M\}$ is a (3,1)-equivalence on M, j=1,2, and the set $\nabla=\{(x,x,y) \mid x,y \in M\}$ is a (3,1)-equivalence, but it is not a (3,2)-equivalence on M. The condition **(M0)** implies that $\Delta \subseteq \rho$.

Definition 2.1. Let $d:M^3 \to R_0^+$ and ρ be as above. If d satisfies **(M0)**, **(P)** and **(Mj)**, $j \in \{1,2\}$, we say that d is a **(3,j,ρ)-metric** on M. If d is a (3,j,Δ)-metric on M, we say that d is a **(3,j)-metric** on M.

Let d be a (3,j,ρ)-metric on M, $j \in \{1,2\}$, $x,y \in M$ and $\varepsilon>0$. As in [6], we consider the following ε-balls, as subsets of M:

a) $B(x,y,\varepsilon)=\{z \mid z \in M, d(x,y,z)<\varepsilon\}$–ε-ball with center at (x,y) and radius ε; and

b) $B(x,\varepsilon)=\{z \mid z \in M$, there is a $v \in M$ such that $d(x,z,v)<\varepsilon\}$– ε-ball with center at x and radius ε.

A (3,j,ρ)-metric d on M induces the following topologies as in [6]:

1) τ(G,d) – the topology generated by all the ε-balls B(x,y,ε), i.e. the topology whose base is the set of the finite intersections of ε-balls B(x,y,ε);

2) τ(D,d) – the topology generated by all the ε-balls B(x,x,ε);

3) τ(N,d) – the topology defined by: $U \in \tau(N,d)$ iff $\forall x \in U$, $\exists \varepsilon>0$ such that $B(x,x,\varepsilon) \subseteq U$; and

4) τ(W,d) – the topology defined by: $U \in \tau(W,d)$ iff $\forall x \in U$, $\exists \varepsilon>0$ such that $B(x,\varepsilon) \subseteq U$.

In [6] we proved that $\tau(W,d) \subseteq \tau(N,d) \subseteq \tau(D,d) \subseteq \tau(G,d)$ for any (3,j,ρ)-metric d.

We order the set $\mathfrak{M}=\{W,N,D,G\}$ by W<N<D<G. For any $X,Y \in \mathfrak{M}$ with X<Y, we say that a topological space (M,τ) is (3,j,ρ)-X-Y-metrizable via a (3,j,ρ)-metric d on M, if $\tau = \tau(X,d) = \tau(Y,d)$. For X=Y, we say only (3,j,ρ)-X-metrizable.

3. SOME SEPARATION PROPERTIES FOR (3,J,ρ)-METRIZABLE SPACES

Proposition 3.1. A (3,1,ρ)-N-metrizable topological space (M,τ), via a (3,1,ρ)-metric d is T_0 space iff for each $a_1\neq a_2\in M$ there is a permutation (x_1,y) of (a_1,a_2) such that for each positive integer n, the following countable many conditions are satisfied:

(*1) there is $\delta_1>0$ such that $y\notin B(x_1,x_1,\delta_1) = C_0$;

(*2) for each $z\in C_0$, there is $\delta_z>0$ such that $y\notin B(z,z,\delta_z)$. Let $C_1 = \cup\{B(z,z,\delta_z) \mid z\in C_0\}$, where $\cup\{B(z,z,\delta_z)|z\in C_0\}$ denotes the union of all the $B(z,z,\delta_z)$, $z\in C_0$;

...

(*n) for each $z\in C_{n-2}$, there is $\delta_z>0$ with $y\notin B(z,z,\delta_z)$. Let $C_{n-1}=\cup\{B(z,z,\delta_z) \mid z\in C_{n-2}\}$;

...

Proof. ⇒: Let (M,τ) be T_0 space, and let $a_1\neq a_2\in M$. W.l.o.g., let $U\in\tau$, such that $a_1=x_1\in U$ and $a_2=y\notin U$. Since (M,τ) is (3,1,ρ)-N-metrizable space, there is $\delta_1>0$ such that $B(x_1,x_1,\delta_1)=C_o\subseteq U$. Since $y\notin U$, it follows that $y\notin C_o$, i.e. (*1). Next, for each $z\in C_o$, since $C_o\subseteq U$, it follows that there is $\delta_z>0$ such that $B(z,z,\delta_z)\subseteq U$. This implies that $y\notin C_1$, i.e. (*2). The same discussion shows that (*n) is satisfied for each positive integer.

⇐: Let $a_1\neq a_2\in M$. W.l.o.g., let $a_1=x_1$ and $a_2=y$. We define $U=\cup\{C_n| n\in N\}$, for the sets from the conditions. The construction of U implies that $U\in\tau$, $x_1\in U$ and $y\notin U$. This shows that (M,τ) is T_0 space. □

Proposition 3.2. A (3,1,ρ)-N-metrizable topological space (M,τ), via a (3,1,ρ) metric d is T_1 space iff $\rho\cap\nabla=\Delta$.

Proof. ⇒: Let (M,τ) be T_1 space. For $x\neq y$, let U_x be an open set such that $x\in U_x$, and $y\notin U_x$. Since U_x is open, there is $\varepsilon>0$ such that $B(x,x,\varepsilon)\subseteq U_x$, and since $y\notin U_x$, we obtain that $d(x,x,y)>\varepsilon$, i.e. $d(x,x,y)\neq 0$. Similarly, $d(x,y,y)\neq 0$. Hence, $\rho\cap\nabla=\Delta$.

⇐: Let $\rho\cap\nabla=\Delta$. Then for each $a\neq b$, $d(a,a,b)\neq 0\neq d(a,b,b)$. Let $x\neq y$. We construct a sequence of sets as in the conditions (*i) from the proof of Proposition 3.1.

(°1) Let $\delta_1=d(x,x,y)$, and let $C_0 = B(x,x,\delta_1)$. Then $y\notin B(x,x,\delta_1)=C_o$.

(°2) For each $z\in C_o$, $z\neq y$. Let $\delta_z=d(z,z,y)$ and let $C_1=\cup\{B(z,z,\delta_z)|z\in C_0\}$. Then $y\notin C_1$.

...

(°n) For each $z\in C_{n-2}$, $z\neq y$. Let $\delta_z=d(z,z,y)$ and let $C_{n-1}=\cup\{B(z,z,\delta_z)|z\in C_{n-2}\}$. Then $y\notin C_{n-1}$.

...

We define $U=\cup\{C_n|\ n\in N\}$. The construction of U implies that $U\in\tau$, $x\in U$ and $y\notin U$. The same construction can be applied to obtain an open set V such that $y\in V$ and $x\notin V$. Hence (M,τ) is T_1-space. □

Proposition 3.2 implies the following corollary.

Corollary 3.1. A $(3,1,\rho)$-N-D-metrizable topological space (M,τ), via a $(3,1,\rho)$ metric d is T_1 space iff $\rho\cap\nabla=\Delta$ **.** □

Proposition 3.3. A $(3,1,\rho)$-W-metrizable topological space (M,τ), via a $(3,1,\rho)$ metric d is T_1 space iff for each $x,y\in M$, $\inf\{d(x,y,z)\mid z\in M\}=0$ implies $x=y$.

Proof. ⇐: Let $x\neq y$. Then $\inf\{d(x,y,z)\mid z\in M\}\neq 0$.

(^{o}1) Let $0<\varepsilon<\inf\{d(x,y,z)|z\in M\}$ and let $C_o=\{z\mid$ there is $u\in M$, $d(x,u,z)<\varepsilon\}$, i.e. let $C_o=B(x,\varepsilon)$. For this C_0, $y\notin C_0$, and $B(x,\varepsilon)\subseteq C_0$.

(^{o}2) For each $u\in C_o$, $u\neq y$, and so $\inf\{d(u,y,z)|z\in M\}\neq 0$. Choose $\varepsilon_u>0$ such that $\varepsilon_u<\inf\{d(u,y,z)|z\in M\}$. Let $C_u=\{z\mid$ there is $v\in M$, $d(u,v,z)<\varepsilon_u\}$, and $C_1=C_0\cup(\cup\{C_u\ |u\in C_o\})$. For this C_1, $y\notin C_1$, and for each $u\in C_0$, $B(u,\varepsilon_u)\subseteq C_1$.

…

(^{o}n+2) Let C_n be defined, such that $C_{n-1}\subseteq C_n$, $y\notin C_n$, and for each $u\in C_{n-1}$, there is $\varepsilon_u>0$ such that $B(u,\varepsilon_u)\subseteq C_n$. For each $u\in C_n$, $u\neq y$, and so $\inf\{d(u,y,z)|z\in M\}\neq 0$. Choose $\varepsilon_u>0$ with $\varepsilon_u<\inf\{d(u,y,z)|z\in M\}$. Let $C_u=\{z\mid$ there is $v\in M$, $d(u,v,z)<\varepsilon_u\}$, and let $C_{n+1}=C_n\cup(\cup\{C_u\ |u\in C_o\})$. For this C_{n+1}, $C_n\subseteq C_{n+1}$, $y\notin C_{n+1}$, and for each $u\in C_n$, $B(u,\varepsilon_u)\subseteq C_{n+1}$.

…

We define $U=\cup\{C_n|n\in N\}$. The construction of U implies that $U\in\tau$, $x\in U$ and $y\notin U$. The same construction can be applied to obtain an open set V such that $y\in V$ and $x\notin V$. Hence (M,τ) is T_1-space.

⇒: Let (M,τ) be T_1 space and let $x\neq y$. Then there is $U\in\tau$, $x\in U$ and $y\notin U$. For this U, there is $\varepsilon>0$, such that $B(x,\varepsilon)\subseteq U$. Since $y\notin U$, it follows that for each $z\in M$, $y,z\notin B(x,\varepsilon)$. This implies that $\inf\{d(x,y,z)|z\in M\}\geq\varepsilon>0$. □

Proposition 3.4. Let (M,τ) be $(3,1,\rho)$-G-metrizable topological space, via a $(3,1,\rho)$ metric d, where $\nabla\subseteq\rho$. Then, for each $x\in M$, the family

$$\{B(x,y_1,\varepsilon)\cap B(x,y_2,\varepsilon)\cap\ldots\cap B(x,y_n,\varepsilon)\}|\ n\in N,\ y_1,y_2,\ldots,y_n\in M,\ \varepsilon>0\}$$

is a local base for the point x.

Proof. Let $x\in B(u,v,\varepsilon)$, $d(u,v,x)=\delta$ and let $\sigma=(\varepsilon-\delta)/2$. If $a\in B(x,u,\sigma)\cap B(x,v,\sigma)$, then $d(u,v,a)\leq d(u,v,x)+d(u,x,a)+d(a,v,x)<\delta+\sigma+\sigma=\varepsilon$, implies that $a\in B(u,v,\varepsilon)$. Since $\nabla\subseteq\rho$, this implies that $x\in B(x,u,\sigma)\cap B(x,v,\sigma)\subseteq B(u,v,\varepsilon)$.

If $x\in V=B(u_1,v_1,\varepsilon_1)\cap B(u_2,v_2,\varepsilon_2)\cap\ldots\cap B(u_n,v_n,\varepsilon_n)$, then there are $\sigma_1,\sigma_2,\ldots,\sigma_n>0$, such that $x\in B(x,u_j,\sigma_j)\cap B(x,v_j,\sigma_j)\subseteq B(u_j,v_j,\varepsilon_j)$. Let $\sigma=\min(\sigma_1,\sigma_2,\ldots,\sigma_n)$. Then

$x\in B(x,u_1,\sigma)\cap B(x,v_1,\sigma)\cap B(x,u_2,\sigma)\cap B(x,v_2,\sigma)\cap\ldots\cap B(x,u_n,\sigma)\cap B(x,v_n,\sigma)\subseteq V$. □

Proposition 3.5. A (3,1,ρ)-G-metrizable topological space (M,τ), via a (3,1,ρ) metric d, where $\nabla\subseteq\rho$ is T_2 space iff for each $x\neq y\in M$ there is $z\in M$ such that d(x,y,z)>0.

Proof. $\Rightarrow$**:** Let (M,τ) be T_2 space and let $x\neq y$. Then there is an open set U such that $y\notin U$. Proposition 3.4., implies that there are $y_1,y_2,\ldots,y_n\in M$ and ε>0, such that $x\in B(x,y_1,\varepsilon)\cap B(x,y_2,\varepsilon)\cap\ldots\cap B(x,y_n,\varepsilon)\subseteq U$. This implies that $y\notin B(x,y_j,\varepsilon)$ for some j, i.e. that $d(x,y_j,y)>\varepsilon>0$.

$\Leftarrow$: Let $x\neq y$, and let z be such that $d(x,y,z)=\varepsilon>0$. Let $U=B(x,y,\varepsilon/3)\cap B(x,z,\varepsilon/3)$ and let $V=B(y,z,\varepsilon/3)$. The assumptions imply that U and V are open sets, $x\in U$ and $y\in V$. Next we will show that $U\cap V=\varnothing$, by contradiction. Let $u\in U\cap V$. Then, $d(x,y,u)<\varepsilon/3$, $d(x,z,u)<\varepsilon/3$, and $d(y,z,u)<\varepsilon/3$. Since d is a (3,1,ρ)-metric, the condition **(M1)** implies that

$$d(x,y,z)\leq d(x,y,u)+d(x,u,z)+d(u,y,z)<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon,$$

a contradiction with $d(x,y,z)=\varepsilon$. □

Proposition 3.6. A (3,1,ρ)-N-D-metrizable topological space (M,τ), via a (3,1,ρ) metric d, is T_0 space iff for each two different points $x,y\in M$, $d(x,x,y)\neq 0$ or $d(x,y,y)\neq 0$.

Proof. $\Rightarrow$: Let (M,τ) be T_0 space and let $x\neq y$. Then there is an open set U such that $x\in U$ and $y\notin U$. This implies that there is an ε>0 such that $B(x,x,\varepsilon)\subseteq U$. Since $y\notin B(x,x,\varepsilon)$, we obtain that $d(x,x,y)>\varepsilon$, i.e. $d(x,x,y)\neq 0$.

$\Leftarrow$: Let (M,τ) be (3,1,ρ)-N-D-metrizable topological space, and let for each two different points $x,y\in M$, $d(x,x,y)\neq 0$ or $d(x,y,y)\neq 0$. Let $d(x,x,y)=\varepsilon>0$. Since for each $x\in M$ and each ε>0 $B(x,x,\varepsilon)\in\tau$, it follows that $x\in B(x,x,\varepsilon)$ and $y\notin B(x,x,\varepsilon)$. Hence, (M,τ) is T_0 space. □

Proposition 3.7. Any (3,j)-D-metrizable space (M,τ), $j\in\{1,2\}$, is T_1 space.

Proof. Let d be a (3,j)-metric, such that $\tau=\tau(D,d)$, let $x,y\in M$ and $x\neq y$. Since d is (3,j)-metric it follows that $d(x,x,y)\neq 0$ and $d(x,y,y)\neq 0$. For $\varepsilon=\min\{d(x,x,y), d(x,y,y)\}$, $x\notin B(y,y,\varepsilon)$ and $y\notin (B,x,x,\varepsilon)$. □

Proposition 3.8. Any (3,2)-D-metrizable space (M,τ) is T_2 space.

Proof. Let d be a (3,2)-metric and $\tau=\tau(D,d)$. Let $x\neq y$ and $d(x,x,y)=\varepsilon$. Let $U_x=B(x,x,\varepsilon/6)$ и $U_y=B(y,y,\varepsilon/6)$. We will show that $U_x\cap U_y=\varnothing$. Conversly, let $U_x\cap U_y\neq\varnothing$ and let $z\in U_x\cap U_y$ then $d(x,x,z)<\varepsilon/6$ and $d(y,y,z)<\varepsilon/6$. Using the fact that d is (3,2)-metric, we have the following inequalities:

$d(x,x,y)\leq 2d(x,z,z)+d(y,z,z)\leq 4d(x,x,z)+2d(y,y,z)<6\varepsilon/6=\varepsilon.$

This contradicts the fact that $d(x,x,y)=\varepsilon$. Hence, $U_x\cap U_y=\varnothing$. □

Proposition 3.9. Any (3,2)-N-metrizable space (M,τ), is T_2 space.

Proof. Let d be a (3,2)-metric and $\tau=\tau(N,d)$. Let $x\neq y$ and $d(x,y,y)=3\varepsilon$. Let $U_x=\{u|d(x,u,u)<\varepsilon\}$ and $U_y=\{|d(y,u,u)<\varepsilon\}$. If $z\in U_x\cap U_y$, i.e. $d(x,z,z)<\varepsilon$ and $d(y,z,z)<\varepsilon$, the condition **(M2)** implies that $d(x,y,y)\leq d(x,z,z)+2d(y,z,z)<3\varepsilon$. This

contradicts the fact that $d(x,y,y)=3\varepsilon$. Hence, $U_x \cap U_y=\emptyset$. Next, let $z\in U_x$ and let $2\delta=\varepsilon-d(x,z,z)$. For $v\in B(z,z,\delta)$ we have that $d(z,z,v)<\delta$ and $d(x,v,v)\leq 2d(z,z,v)+d(x,z,z)<2\delta+\varepsilon=\varepsilon$. Hence, $B(z,z,\delta)\subseteq U_x$. This implies that U_x is open, i.e. $U_x\in\tau(N,d)$. Similarly, U_y is open. All this shows that (M,τ) is T_2 space. □

In [6] we proved that any (3,2)-N-D-metrizable space (M,τ) is metrizable, so by this we have the following corollary.

Corollary 3.2. Any (3,2)-N-D-metrizable space (M,τ) is perfectly normal. □

Next we will show that any (3,2)-N-metrizable space is perfectly normal.

Proposition 3.10. Any (3,2)-N-metrizable space (M,τ) is perfectly normal.

Proof. Let d be a (3,2)-metric on M such that $\tau=\tau(N,d)$. It is shown in [12], that for any $A\subseteq M$, the map $f_A:M\to R_0^+$ defined by $f_A(x)=\inf\{d(x,x,a)|a\in A\}$, is continuous. For $x\in A$, $f_A(x)=0$. Moreover, if A is closed and $x\notin A$, then $f_A(x)\neq 0$. Next, let A, B be two disjoint closed subsets of M. Since f_A, f_A+f_B are continuous functions, and for any x, $f_A(x)+f_B(x)\neq 0$, it follows that $f:M\to R_0^+$, defined by $f(x)=f_A(x)/(f_A(x)+f_B(x))$ is continuous. Moreover, $f(x)=0$ if and only if $x\in A$, and $f(x)=1$ if and only if $x\in B$. All this shows that (M,τ) is perfectly normal. □

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A Characterization of Generalized $(m+k,m)$-Rectangular Bands

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Abstract: *The definition and a structural description of the generalized $(m+k,m)$-rectangular bands are given in [6]. In this paper we give a characterization of generalized $(m+k,m)$-rectangular bands.*

Keywords: *$(m+k,m)$-rectangular band, generalized $(m+k,m)$-rectangular band.*

1. INTRODUCTION

We introduce some notations which will be used further on:

1) The elements of Q^s, where Q^s denotes the $s-$th Cartesian power of Q, will be denoted by x_1^s.
2) The symbol x_i^j will denote the sequence $x_i x_{i+1}...x_j$ when $i \le j$, and the empty sequence when $i > j$.
3) If $x_1 = x_2 = ... = x_s = x$, then x_1^s is denoted by the symbol $\overset{s}{x}$.
4) The set $\{1,2,...,s\}$ will be denoted by $\mathbf{N_s}$.

Let $Q \neq \emptyset$ and n,m be positive integers. We say that a map $[\ ]$ from Q^n to Q^m, is an (n,m)-operation. A pair $(Q;[\ ])$ where $[\ ]$ is an (n,m)-operation is called an (n,m)-groupoid. Every (n,m)-operation on Q induces a sequence $[\ ]_1,[\ ]_2,...,[\ ]_m$ of n-ary operations on the set Q, such that

$$((\forall i \in \mathbf{N_m})\ \left[x_1^n\right]_i = y_i) \Leftrightarrow \left[x_1^n\right] = y_1^m.$$

Let $m \ge 2, k \ge 1$. An $(m+k,m)$-groupoid $(Q;[\ ])$ is called an $(m+k,m)$-semigroup if for each $i \in \{0,1,2,...,k\}$

$$\left[x_1^i \left[x_{i+1}^{i+m+k}\right] x_{i+m+k+1}^{m+2k}\right] = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right].$$

Let $(L;[\ ])$ be an $(m+k,m)$-groupoid, where $[\ ]$ is an $(m+k,m)$-operation defined by $[x_1^{m+k}] = x_1^m$. Then $(A;[\ ])$ is an $(m+k,m)$-semigroup and it is called

a left-zero $(m+k,m)$-semigroup. Dually, a right-zero $(m+k,m)$-semigroup $(R;[\ ])$ is defined by the operation $[x_1^{m+k}]=x_{k+1}^{m+k}$.

The pair $(L\times R;[\ ])$, where $[\ ]$ is an $(m+k,m)$-operation on $L\times R$ defined by

$$[x_1^{m+k}]=y_1^m \ \Leftrightarrow\ (x_i=(a_i,b_i), y_j=(a_j,b_{j+k}), i\in \mathbf{N_{m+k}},\ j\in \mathbf{N_m})$$

is an $(m+k,m)$-semigroup and it is a direct product of a left-zero and a right-zero $(m+k,m)$-semigroups L and R. Such an $(m+k,m)$-semigroup is called an $(m+k,m)$-rectangular band.

The next proposition gives a characterization of $(m+k,m)$-rectangular bands as $(m+k,m)$-semigroups in which three identities are satisfied ([3], [4]).

Proposition 1.1. Let $\mathbf{Q}=(Q;[\ ])$ be an $(m+k,m)$-semigroup. $\mathbf{Q}$ is an $(m+k,m)$-rectangular band if and only if for all $i,j\in \mathbf{N_m}$, the following identities are satisfied in $\mathbf{Q}$:

(RB I) $\left[x_1^{m+k}\right]_i=\left[y_1^{j-1}x_i y_{j+1}^{j+k-1}x_{i+k}y_{j+k+1}^{m+k}\right]_j$,

(RB II) $\left[x_1^{m+2k}\right]_i=\left[x_1^i x_{i+k+1}^{m+2k}\right]_i$,

(RB III) $\left[\overset{m+k}{x}\right]=\overset{m}{x}$.

A generalization of $(m+k,m)$-rectangular band is made by omitting the identity (RB III), i.e. an $(m+k,m)$-semigroup $(Q;[\ ])$ in which identities (RB I) and (RB II) are satisfied is called a generalized $(m+k,m)$-rectangular band ([6, Definition 2.1.]).

Next we will give a structural description of generalized $(m+k,m)$-rectangular bands.

Let O and R be two non empty disjoint sets, and let $(R;[\ ])$ be an $(m+k,m)$-rectangular band. Let $Q=R\cup O$ and $\psi:O\to R$ be an arbitrary map. We extend ψ to the map $\varphi:Q\to R$ defined by $\varphi(x)=\begin{cases}\psi(x), & x\in O\\ x, & x\in R\end{cases}$, and we define an $(m+k,m)$-operation on Q, by $\left[x_1^{m+k}\right]=[\varphi(x_1)\varphi(x_2)...\varphi(x_{m+k})]$. An $(m+k,m)$-semigroup $\mathbf{Q}=(Q;[\ ])$ obtained by this construction, will be denoted by $\mathbf{Q}=[R,O,\psi]$.

Theorem 1.2. ([6, Theorem 2.4.]) An $(m+k,m)$-semigroup $\mathbf{Q}=(Q;[\ ])$ is a generalized $(m+k,m)$-rectangular band that is not an $(m+k,m)$-

rectangular band if and only if $\mathbf{Q} = [R, O, \psi]$, for some $(m+k, m)$-rectangular band $(R;[\ ])$, nonempty set O disjoint from R, and a map $\psi : O \to R$.

2. A CHARACTERIZATION OF GENERALIZED $(m+k, m)$-RECTANGULAR BANDS

In the sequel we will give a characterization of a generalized $(m+k, m)$–rectangular band, using a rectangular band I, i.e. a semigroup $(Q;*)$ that satisfy the identity $x*y*z = x*z$, for each x, y, $z \in Q$.

Theorem 2.1. $\mathbf{Q} = (Q;[\ ])$ is a generalized $(m+k, m)$-rectangular band if and only if there is a rectangular band I, $(Q;*)$, such that $\left[x_1^{m+k}\right]_i = x_i * x_{i+k}$, $x_1^{m+k} \in Q^{m+k}$, $i \in \mathbf{N_m}$.

Proof. Let $\mathbf{Q} = (Q;[\ ])$ be a generalized $(m+k, m)$-rectangular band, i.e. (RB I) and (RB II) are satisfied in $\mathbf{Q}$. Let $a \in Q$ be fixed and let $*$ be the operation on Q defined by $x*y = \left[\begin{smallmatrix} k-1 & m-1 \\ x\ a\ y\ a \end{smallmatrix}\right]_1$. Clearly, $(Q;*)$ is a groupoid.

Let $x, y, z \in Q$. Then:

$$(x*y)*z = \left[\left[\begin{smallmatrix} k-1 & m-1 \\ x\ a\ y\ a \end{smallmatrix}\right]_1 \begin{smallmatrix} k-1 & m-1 \\ a\ z\ a \end{smallmatrix}\right]_1 \overset{\mathbf{RB\,I}}{=} \left[\begin{smallmatrix} m-1 \\ a \end{smallmatrix}\left[\begin{smallmatrix} m-1 & k-1 \\ a\ x\ a\ y \end{smallmatrix}\right]_m \begin{smallmatrix} k-1 \\ a\ z \end{smallmatrix}\right]_m$$

$$\overset{\mathbf{RB\,I}}{=} \left[\left[\begin{smallmatrix} m-1 & k-1 \\ a\ x\ a\ y \end{smallmatrix}\right]_1 \ldots \left[\begin{smallmatrix} m-1 & k-1 \\ a\ x\ a\ y \end{smallmatrix}\right]_{m-1} \left[\begin{smallmatrix} m-1 & k-1 \\ a\ x\ a\ y \end{smallmatrix}\right]_m \begin{smallmatrix} k-1 \\ a\ z \end{smallmatrix}\right]_m = \left[\begin{smallmatrix} m-1 & k-1 & k-1 \\ a\ x\ a\ y\ a\ z \end{smallmatrix}\right]_m$$

$$\overset{\mathbf{RB\,II}}{=} \left[\begin{smallmatrix} m-1 & k-1 \\ a\ x\ a\ z \end{smallmatrix}\right]_m \overset{\mathbf{RB\,I}}{=} \left[\begin{smallmatrix} k-1 & m-1 \\ x\ a\ z\ a \end{smallmatrix}\right]_1 \overset{\mathbf{RB\,II}}{=} \left[\begin{smallmatrix} k-1 & k-1 & m-1 \\ x\ a\ y\ a\ z\ a \end{smallmatrix}\right]_1$$

$$= \left[\begin{smallmatrix} k-1 \\ x\ a \end{smallmatrix}\left[\begin{smallmatrix} k-1 & m-1 \\ y\ a\ z\ a \end{smallmatrix}\right]_1 \left[\begin{smallmatrix} k-1 & m-1 \\ y\ a\ z\ a \end{smallmatrix}\right]_2 \ldots \left[\begin{smallmatrix} k-1 & m-1 \\ y\ a\ z\ a \end{smallmatrix}\right]_m\right]_1 \overset{\mathbf{RB\,I}}{=} \left[\begin{smallmatrix} k-1 \\ x\ a \end{smallmatrix}\left[\begin{smallmatrix} k-1 & m-1 \\ y\ a\ z\ a \end{smallmatrix}\right]_1 \begin{smallmatrix} m-1 \\ a \end{smallmatrix}\right]_1$$

$$= x*(y*z)$$

So, $(Q;*)$ is a semigroup.

$(Q;*)$ is a rectangular band I because $x*y*z = \left[\begin{smallmatrix} k-1 & m-1 \\ x\ a\ z\ a \end{smallmatrix}\right]_1 = x*z$.

Moreover, we have

$$\left[x_1^{m+k}\right]_i \overset{\mathbf{RBI}}{=} \left[x_i \overset{k-1}{a} x_{i+k} \overset{m-1}{a}\right]_1 = x_i * x_{i+k},\ x_1^{m+k} \in Q^{m+k},\ i \in \mathbf{N_m}.$$

Conversely, let $(Q;*)$ be a rectangular band I. We define $[\ ]: Q^{m+k} \to Q^m$ by $\left[x_1^{m+k}\right]_i = x_i * x_{i+k}$.

Clearly, $(Q;[\ ])$ is an $(m+k,m)$-groupoid.

In order to prove that $(Q;[\ ])$ is an $(m+k,m)$-semigroup, we need to go through the following three cases:

1) $k = m$;

2) $k > m$, i.e. $k = m+s,\ s \geq 1$ and

3) $k < m$, i.e. $k+t = m,\ t \geq 1$.

1) $\left[\left[x_1^{2m}\right]x_{2m+1}^{3m}\right]_i = \left[x_1^{2m}\right]_i * x_{2m+i} = (x_i * x_{i+m}) * x_{2m+i} = x_i * x_{2m+i}$.

We will prove that $\left[x_1^j\left[x_{j+1}^{j+2m}\right]x_{j+2m+1}^{3m}\right]_i = x_i * x_{2m+i},\ j \in \mathbf{N_m}$.

a) Let $i \leq j$. Then $i+m \leq j+m$. We obtain $i+m = j+\lambda$ for $\lambda \in \mathbf{N_m}$. Then

$$\left[x_1^j\left[x_{j+1}^{j+2m}\right]x_{j+2m+1}^{3m}\right]_i = x_i * \left[x_{j+1}^{j+2m}\right]_\lambda = x_i * \left(x_{j+\lambda} * x_{j+\lambda+m}\right) = x_i * x_{j+\lambda+m}$$

$$= x_i * x_{i+m+m} = x_i * x_{2m+i}.$$

b) Let $j < i$. Because $j < m$, we obtain $j+r = m,\ j+\lambda = i$ where $\lambda \in \mathbf{N_r}$ and $i+m > j+m$. Then:

$$\left[x_1^j\left[x_{j+1}^{j+2m}\right]x_{j+2m+1}^{3m}\right]_i = \left[x_{j+1}^{j+2m}\right]_\lambda * x_{j+2m+\lambda} = \left(x_{j+\lambda} * x_{j+\lambda+m}\right) * x_{j+2m+\lambda}$$

$$= x_{j+\lambda} * x_{j+2m+\lambda} = x_i * x_{2m+i}.$$

Hence, $(\forall i, j \in \mathbf{N_m})\ \left[\left[x_1^{2m}\right]x_{2m+1}^{3m}\right]_i = \left[x_1^j\left[x_{j+1}^{j+2m}\right]x_{j+2m+1}^{3m}\right]_i$, i.e. $(Q;[\ ])$ is a $(2m,m)$-semigroup.

2) $\left[\left[x_1^{2m+s}\right]x_{2m+s+1}^{3m+2s}\right]_i = \left[x_1^{2m+s}\right]_i * x_{2m+s+i+s} = (x_i * x_{i+m+s}) * x_{2m+2s+i} = x_i * x_{2m+2s+i}$.

We will prove that $\left[x_1^j\left[x_{j+1}^{j+2m+s}\right]x_{j+2m+s+1}^{3m+2s}\right]_i = x_i * x_{2m+2s+i}$, for $j \in \mathbf{N_{m+s}}$.

a) Let $i < j(\leq m+s)$.

a1) Let $j \leq s$. Then $i+m+s > s+m \geq j+m$ and

$\left[x_1^j\left[x_{j+1}^{j+2m+s}\right]x_{j+2m+s+1}^{3m+2s}\right]_i = x_i * x_{j+2m+s+i+s-j} = x_i * x_{2m+2s+i}$.

a2) If $s < j \leq m+s$, then $s+r = j, r \geq 1$. $s+r \leq m+s$ implies $r \in \mathbf{N_m}$.

a2.1) Let $i \leq r$. Then $i+m+s \leq r+m+s = j+m$ and

$\left[x_1^j\left[x_{j+1}^{j+2m+s}\right]x_{j+2m+s+1}^{3m+2s}\right]_i$

$= \left[x_1^{s+r}\left[x_{j+1}^{j+2m+s}\right]_1 \ldots \left[x_{j+1}^{j+2m+s}\right]_{m-r}\left[x_{j+1}^{j+2m+s}\right]_{m-r+1} \ldots \left[x_{j+1}^{j+2m+s}\right]_{m-r+r} x_{j+2m+s+1}^{3m+2s}\right]_i$

$= x_i * \left[x_{j+1}^{j+2m+s}\right]_{m-r+i} = x_i * \left(x_{j+m-r+i} * x_{j+m-r+i+m+s}\right) = x_i * x_{j+m-r+i+m+s}$

$= x_i * x_{s+r+2m-r+i+s} = x_i * x_{2m+2s+i}$.

a2.2) Let $i > r$. Then $i+m+s > r+m+s = j+m$ and

$\left[x_1^j\left[x_{j+1}^{j+2m+s}\right]x_{j+2m+s+1}^{3m+2s}\right]_i = x_i * x_{j+2m+s+i-r} = x_i * x_{s+r+2m+s+i-r} = x_i * x_{2m+2s+i}$.

b) Let $i = j$. Then $\left[x_1^j\left[x_{i+1}^{i+2m+s}\right]x_{i+2m+s+1}^{3m+2s}\right]_i = x_i * x_{i+2m+s+s} = x_i * x_{2m+2s+i}$.

c) Let $i > j$. Then, $i \leq m$ implies that $j < m$. Let $j+r = m$. Since $j+1 \leq i \leq j+r$, let $i = j+\lambda$ where $\lambda \in \mathbf{N_r}$. Then

$\left[x_1^j\left[x_{j+1}^{j+2m+s}\right]x_{j+2m+s+1}^{3m+2s}\right]_i = \left[x_{j+1}^{j+2m+s}\right]_\lambda * x_{j+2m+s+s+\lambda}$

$= \left(x_{j+\lambda} * x_{j+\lambda+m+s}\right) * x_{2m+2s+i} = x_{j+\lambda} * x_{2m+2s+i} = x_i * x_{2m+2s+i}$.

Hence, $(\forall i \in \mathbf{N_m},\ j \in \mathbf{N_{m+s}})\ \left[\left[x_1^{2m+s}\right]x_{2m+s+1}^{3m+2s}\right]_i = \left[x_1^j\left[x_{j+1}^{j+2m+s}\right]x_{j+2m+s+1}^{3m+2s}\right]_i$, i.e. $(Q;[\ ])$ is a $(2m+s, m)$-semigroup, $s \geq 1$.

3) First, we will prove that $\left[\left[x_1^{m+k}\right]x_{m+k+1}^{m+2k}\right]_i = x_i * x_{2k+i}$, for each $i \in \mathbf{N_m}$.

a) Let $i \leq t$. Then $i+k \leq t+k = m$ and

$\left[\left[x_1^{m+k}\right]x_{m+k+1}^{m+2k}\right]_i = \left[x_1^{m+k}\right]_i * \left[x_1^{m+k}\right]_{i+k} = \left(x_i * x_{i+k}\right) * \left(x_{i+k} * x_{i+2k}\right) = x_i * x_{2k+i}$.

b) Let $t < i \le m = t + k$. Then $i = t + \lambda$ where $\lambda \in \mathbf{N_k}$ and $i + k = t + \lambda + k = m + \lambda$. Then

$$\left[\left[x_1^{m+k}\right]x_{m+k+1}^{m+2k}\right]_i = \left[x_1^{m+k}\right]_i * x_{m+k+\lambda} = (x_i * x_{i+k}) * x_{i+k+k} = x_i * x_{2k+i}.$$

Further on, we will prove that $\left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i = x_i * x_{2k+i}$, for each $i \in \mathbf{N_m}$ and each $j \in \mathbf{N_k}$.

c) Let $i \le j$. Then $i \le j < j + t$ implies $i + k < j + t + k = j + m$. Because $i + k > k \ge j$, we obtain $j < i + k < j + m$. Let $i + k = j + \lambda$. We have

$$\left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i =$$

$$\left[x_1^j\left[x_{j+1}^{j+m+k}\right]_1 \ldots \left[x_{j+1}^{j+m+k}\right]_\lambda \left[x_{j+1}^{j+m+k}\right]_{\lambda+1} \ldots \left[x_{j+1}^{j+m+k}\right]_m x_{j+m+k+1}^{m+2k}\right]_i$$

$$= x_i * \left[x_{j+1}^{j+m+k}\right]_\lambda = x_i * \left(x_{j+\lambda} * x_{j+\lambda+k}\right) = x_i * x_{j+\lambda+k} = x_i * x_{i+k+k} = x_i * x_{2k+i}.$$

d) Let $j < i$.

d1) Let $i = j + \lambda$, $\lambda \in \mathbf{N_t}$. Then $i + k = j + \lambda + k \le j + t + k = j + m$ and

$$\left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i = \left[x_{j+1}^{j+m+k}\right]_\lambda * \left[x_{j+1}^{j+m+k}\right]_{\lambda+k}$$

$$= \left(x_{j+\lambda} * x_{j+\lambda+k}\right) * \left(x_{j+\lambda+k} * x_{j+\lambda+k+k}\right) = x_{j+\lambda} * x_{j+\lambda+2k} = x_i * x_{2k+i}.$$

d2) If $j + t < i$ then $i = j + t + \lambda$, $\lambda \in \mathbf{N_{k-j}}$. Then

$i + k = j + t + \lambda + k = j + m + \lambda > j + m$. We have

$$\left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i = \left[x_{j+1}^{j+m+k}\right]_{t+\lambda} * x_{j+m+k+\lambda} = \left(x_{j+t+\lambda} * x_{j+t+\lambda+k}\right) * x_{j+m+k+\lambda}$$

$$= x_{j+t+\lambda} * x_{j+m+k+\lambda} = x_i * x_{j+k+t+k+\lambda} = x_i * x_{2k+i}.$$

Hence, $(\forall i \in \mathbf{N_m},\ j \in \mathbf{N_k})\ \left[\left[x_1^{m+k}\right]x_{m+k+1}^{m+2k}\right]_i = \left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i$, i.e. $(Q;[\ ])$ is an $(m+k, m)$-semigroup, $k < m$.

Let $x_1^{m+k}, y_1^{m+k} \in Q^{m+k}$ and $x_i = y_j$, $x_{i+k} = y_{j+k}$ for some $i, j \in \mathbf{N_m}$.

Then $\left[x_1^{m+k}\right]_i = x_i * x_{i+k} = y_j * y_{j+k} = \left[y_1^{m+k}\right]_j = \left[y_1^{j-1} x_i y_{j+1}^{j+k-1} x_{i+k} y_{j+k+1}^{m+k}\right]_j$.

So, $(Q;[\ ])$ satisfies (RB I).

Let $x_1^{m+2k} \in Q^{m+2k}$, $i \in \mathbf{N_m}$. Then $\left[x_1^{m+2k}\right]_i = x_i * x_{2k+i} = \left[x_1^i x_{i+k+1}^{m+2k}\right]_i$.

So, $(Q;[\ ])$ satisfies (RB II).

Hence, $(Q;[\ ])$ is a generalized $(m+k,m)$-rectangular band. ■

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Optimization Model of Loan Portfolio

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***Abstract**: Economic and mathematical model for the credit management of commercial banks optimization problem is developed. The model allows obtaining optimal strategies of variant calculations for this kind of banking services. The procedure of optimal solution of this problem is designed to help the decisions making concerning the optimal allocation of assets. The aim of research is to analyze current trends in the credit market and find ways to optimize the structure of credit portfolio of banks.*

***Keywords:** bank, deposit, credit portfolio, model, optimal management;*

1. INTRODUCTION

Successful functioning of the bank depends not only on workers' skills, their knowledge and experience, but also on the optimization of the whole process of making and implementation of management decisions.

Modelling of bank financial activity is very important and difficult task, since the bank is a system in which deterministic and random processes take place simultaneously and they are interrelated by very complicated factors. Furthermore, subjective management decisions are essential in bank activity. However, a bank interacts with the financial markets in the processes of functioning, which are difficult to model. All these facts point at considerable complexity of creating an analytical model of bank financial activity, which can be used in practice.

H. Markowitz, W. Sharpe, E. Baltensperger, J. Franke, [2, 5, 6] and others have worked on the problem of modelling of optimal credit portfolio. One of the most important tasks of commercial bank in the field of financial activity is to balance between profitability and risk and to find optimal combination in the form of tradeoffs.

The aim of research is to reveal current practices of the credit portfolio and to analyze current trends in the credit market. On this base is created a model for loan portfolio optimization and ways to optimize the structure of credit portfolio of banks are searched for.

2. ANALYSIS OF CURRENT TRENDS

The modern trends in the management of credit portfolio risks require special techniques for data processing in order to create an analytical basis for management decisions. One of the solutions to the problem of maximizing profit from lending with a minimum level of risk is the application of linear programming.

The structure of credit portfolio of banks of Bulgaria is characterized by several negative factors: growth is very small, which is a sign of decreasing domestic demand, loan portfolio for entities increases the minimum degree, interest rates on funds of individuals are at a maximum of the year, a large proportion of bad loans in the portfolio.

This makes the urgency of measures to optimize the banks' loans structure, which provide the development of effective risk management and use mathematical programming in their activities.

Based on the classic approach of modelling of credit portfolio, we model optimal structure of commercial bank loan portfolio.

Let be considered data about a real life bank portfolio. According to the Bulgarian National Bank in 2013 the share of problem loans in the portfolios of banks has increased in 1.02 times [3]. In 2014, asset qualities continued to decrease – according to the information of BNB 16.44% of all loans were problematic in June, 2014. In this regard, analysis of optimizing the structure of loan portfolio is of particular significance that leads to urgency and expediency of the chosen research topic for development of algorithms and methods in this area.

The review of the banks' loan portfolio segments showed that the performance of corporate loans was again the most problematic, and since June 30th 2014 the share of the loans that were more than 90 days overdue (including those that were more than 180 days overdue) was amounting to 18.5%. The share of the household loans having the same delay was 13.4%. The overdue share of the residential mortgage loans was amounting to 17.0%, while the level of the overdue consumer loans (9.6%) remained the most favorable [4].

3. THE PROPOSED MODEL

Credit risk is one of the most important banking risks; in addition, it becomes a cause for nonperforming loans and losses related to the insolvency of the borrower [1].

To become effective the risk-management, any bank should have effective credit risk management system, which should solve the following tasks:

1) forming characteristics of the borrower (issuer rating and the probability of default);

2) determining the share of problem loans through restructuring and sails of debts;

3) verification of the terms of agreements and decisions on lending;

4) increasing competitive advantages by improving the quality of loan portfolio;

5) possibility of permanent control over the portfolio;

6) increase discipline and reduce time expenses due to standardization and automation;

7) opportunities for ongoing monitoring and fast reaction at client's problems.

Commercial bank may carry out credit, investment and other active operations only within available financial resources. Deposits are principal source of financial resources of commercial bank: they determine the scope and range of profitable operations of the bank.

The main management instrument of credit activity is economic and mathematical methods and models.

The following conditions are prerequisite of building economic and mathematical model of a bank loan portfolio:

- the timing of the credit arrangement;
- the calculation of annual and monthly crediting rates;
- the determination of the conditions for loan repayment;
- the determination of the home equity sum that can be used for the crediting process;
- the determination of the risk magnitude as the default probability of all types of credits and the calculation the risk of default by periods;
- the use of current percent of credit repayments in the crediting process;
- the determination of the period, at the beginning of which the calculation of profits is carried out;
- Taking into account the liquidity ratios (H4, H5 and H6) that regulate the credit activity of a commercial bank.

We denote the scope of granted credits through x_{ij} ($i_t = \overline{1,n}$; n=18), t_i – a period of time – a month for the i-th type of credit $t_i \in T_i$, T_i – the set of periods for the i-type credit, a quarter: a_{ij} – a matrix that shows the process of providing the i-type credit in the period t or the lack of it:

$$a_{it_1} = \begin{cases} 1, \text{ if } i\text{-type credit is given in period } t_i; \\ 0, \text{ if } i\text{-type credit is not given in period } t_i; \end{cases}$$

Highly liquid investments of customers – demand deposits – are used in order to provide short-term liquidity for lending short-term credits (up to 1 month).

Bank funds and term deposits are used for lending other types of credits.

The bank's ability to pay its current obligations from the assets of the primary and secondary liquidity is determined by the current liquidity H5, which value for

a given period for commercial banks is 0.4. At present, current obligations of the bank can be more than 2.5 times bigger than its current assets.

In order to organize the efficient crediting process with the help of rapid accumulation of funds, credits with the same expiration dates have different interest rates and different conditions of return. Some are traditionally paid, i.e. monthly interest, and the principle amount of the credit is at the end of the term: the others are returned with monthly interest and part of the principle amount of the credit, which is defined as following. Principle amount of the credit is divided into equal parts for the whole credit term.

We accept that funds received from loan interest of the previous period, and refunded credits are used for credit arrangement in the next period.

We introduce the following notation: $\boldsymbol{r}_i$-monthly interest rate for i-type credit; t_i-the period for which the i-type credit is granted; S_t – the amount of payment received from credits granted with special conditions in the period t; SK – bank stock; p_i – default probability of i-type credit; u_{it} – the risk value of default of i-type credit in the period t, it is calculated as following: $t = 1$, $u_{it}=p_i$; $t = 2$, $u_{i2}–p_i.(1–p)$; $t = 3$, $u_{i3}–p_i.(1–p)^2$; D_{jt} – the amount of available deposit funds of j-type in the period t, m – the number of deposit types on lead time on investment; z_j– monthly percentage rate on deposits payment of j-type.

Taking into account the conditions described above and introduced notation, we form the economic and mathematical model of the problem.

The objective function of the problem is to maximize the highest income yield at the beginning of the next period:

$$Z = \sum_{t_2=1}^{\tau_2} \sum_{i=1}^{n} r_i x_{it} - \sum_{t=1}^{T} \sum_{j=1}^{m} z_j D_{jt} \rightarrow \max, t_i \in T_i \quad (1)$$

subject to the following constraints:

Balance conditions in the primary period of crediting process $t = 1$

$$\sum_{i=1}^{n} x_{i1} = SK + \sum_{j=1}^{m} D_{j1} \quad (2)$$

Balance conditions in the secondary period of crediting process $t = 2$

$$\sum_{i=1}^{n} x_{i2} + \sum_{j=1}^{m} z_j D_{j1} + D_1 \le \sum_{j=1}^{m} D_{j2} + \sum_{i=1}^{n} r_i x_{i1} + S_1 \quad (3)$$

$$S_1 = x_{1,1} + \frac{x_{1,3}}{\tau_3} + \frac{x_{1,5}}{\tau_5} + \frac{x_{1,7}}{\tau_7} + \frac{x_{1,9}}{\tau_9} + x_{10,1} + \frac{x_{12,1}}{\tau_{12}} + \frac{x_{14,1}}{\tau_{14}} + \frac{x_{16,1}}{\tau_{16}} + \frac{x_{18,1}}{\tau_{18}} \quad (4)$$

Balance conditions in the period of crediting process $t = 3$

$$S_2 = x_{1,2} + 2.\left(\frac{x_{1,3}}{\tau_3} + \frac{x_{1,5}}{\tau_5} + \frac{x_{1,7}}{\tau_7} + \frac{x_{1,9}}{\tau_9} \right) + x_{10,2} + 2.\left(\frac{x_{12,1}}{\tau_{12}} + \frac{x_{14,1}}{\tau_{14}} + \frac{x_{16,1}}{\tau_{16}} + \frac{x_{18,1}}{\tau_{18}} \right) \quad (5)$$

Constraints for short-term and highly liquid credits:

$$x_{1,1} + x_{10,1} \ge D_{1,1} \quad ; \quad x_{1,2} + x_{10,2} \ge D_{1,2} \quad ; \quad x_{1,3} + x_{10,3} \ge D_{1,3} \quad (6)$$

Constraints for the scope use of borrowed funds in the total amount of credits:

$$\frac{\sum_{j=1}^{m}\sum_{t=1}^{3} D_{j,t}}{\sum_{i=1}^{n}\sum_{t=1}^{3} x_{it}} < 1; \tag{7}$$

Constraints accounting the implementation of current liquidity standards by periods:

$$\sum_{i=1}^{18} x_{i,1} \geq 0,4\sum_{j=1}^{5} D_{j,1} \quad ; \quad \sum_{i=1}^{18} x_{i,2} \geq 0,4\sum_{j=1}^{5} D_{j,2} \quad ; \quad \sum_{i=1}^{18} x_{i,3} \geq 0,4\sum_{j=1}^{5} D_{j,3} \tag{8}$$

Constraints for size of average risk of credit defaults:

$$\frac{\sum_{i=1}^{n}\sum_{t_i=1}^{T_i} u_{it} - x_{it}}{\sum_{i=1}^{n}\sum_{t_1=1}^{T_i} x_{it_1}} \leq U^{*} \tag{9}$$

Constraints that determine the marginal sum of volume of credit:

$$\sum_{i=1}^{n}\sum_{t_1=1}^{T_i} x_{it_1} \leq K_0 \tag{10}$$

Constraints for nonnegative variables:

$$x_{it_i} \geq 0, i = \overline{1,n};\ t_i = \overline{1,T}_i \tag{11}$$

4. CONCLUSION

In 2014-2015 a further deterioration could be expected due to the bankruptcy of the CTB, but not as a result of a real worsening of the ratio between performing and non-performing loans. The growth of the non-performing loans will be caused by adjustments due to removing the CTB's data out of the statistics and adding those bad loans to the data of other banks.

The presented mathematical model enables obtaining the optimal scheme of the crediting process, which is divided monthly, that allows us to trace the funds movement, calculate idle balances and direct them to acquire marketable instruments, provide short-term interbank credits in another bank. Since the model is designed for one quarter (3 months), bankers can summarize their activities per quarter, and make adjustments in the subsequent period. It should be noted that the optimization model is quite flexible. Additional constraints could be added to it, to model the current situation of credit and deposit activities. For example, they may be the constraints accounting known volume of credits at the beginning of the crediting period, the current values, and necessary reserve

funds. If the desirable amount of profit is noted in the target function, then we obtain necessary scheme of credit allocation for periods and types. In addition, if necessary, you can change the amount of the deposit incomes. We use the EXEL package to obtain numerical solutions of constructed model. Optimization algorithm indicates the direction of the search; conjugate gradient method [7] is selected, which is used to solve big optimization problems. During the solving of the problem with this search method, the large number of iterations is performed; this gives the possibility to obtain accurate results. At the end of the period maximum profit and the scheme of credit arrangement on volume, types and periods is obtained. The optimization algorithm will be tested on real life portfolio optimization problems.

Quantitative analysis of different scenarios provides a selection of profitable option for the bank, the estimate of probability of a bad situation and makes it possible to develop an adequate plan of action. Using simulation models one can evaluate and analyze alternative scenarios of profit and risk management. The banks could develop their own models that take into account the specificity of their activity or use already created ones.

5. ACKNOWLEDGEMENTS

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Solution to Discrete Exterior Dirichlet Problem of Gravitational Potential Determination to Real Geometries

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Abstract: *There are known solutions refering to spherical and ellipsoidal Earth models. However, the newest technological achievments require a more realistic approach in solving this problem considering the real Earth surface and the real Earth body. In this paper we present a solution to this problem with respect to the real Earth surface. This solution overcomes the main drawback of the solutions in the spherical case, meaning the approximating spline has the same domain of harmonicity as the real gravitational potential.*

Keywords: *gravitational potential, boundary value problem, reproducing kernel, harmonicity.*

1. INTRODUCTION

In gravitational field theory, the relation between the object function, i.e., the gravitational potential V and the data is non-linear. However, it may supposed to be linear if we go over from the gravitational potential to a (suitably defined) anomalous gravitational potential. Mathematically, the handling of the anomalous gravitational potential is equivalent to restricting the gravitational field theory to a linear relation of the object function to the data. To be more concrete, the actual problem of gravitational theory today is the determining of a harmonic function, regular at infinity, to certain linear functionals, for example, discrete boundary data on the Earth's surface or discrete satellite data from space. In consequence, gravitational field theory canonically leads to interpolation based on a specific linear functionals, usually functional values or derivatives in certain (discretely given) points. In the conventional geodetic approach it was proposed, that the class of approximating functions should conveniently be structured as a Hilbert space with reproducing kernel. There are several reasons for using this topological structure:

- In accordance with the fact that the Laplace operator is a linear differential operator, the gravitational potential can be obtained by superposition of certain potential functions.
- By introducing the norm in a reproducing Hilbert space, it is easily possible to specify the class of approximating functions and to control the accuracy of the approximation.
- All linear (observation) functionals of terrestrial, airborne as well as spaceborne origin can be identified with elements of the dual space of this Hilbert space.
- Reproducing kernel functions turn out to have extremely desirable properties as interpolating, smoothing, and best approximation functions.
- The problem of interpolation using reproducing kernels becomes inextricably involved with the problem of choosing a specific norm [2]. This exposes the strength and the weakness of the method of interpolation. Given a norm in a Hilbert space we can calculate the reproducing kernel (if it exists), which again delivers the interpolating function under minimum norm assumptions.

2. RKHS OF NEWTONIAN POTENTIALS

Note that from now on the Earth surface will be understood as a regular surface Σ, and it's interior and exterior space, respectively is denoted by Σ^{int} and Σ^{ext}. In Newtonian nomenclature, the gravitational potential V of the Earth generated by a mass-distribution F inside the Earth is given by the volume integral (Newton integral)

$$(1) \qquad V(x) = \int_{\Sigma^{int}} \frac{F(y)}{|x-y|} dy, \quad x \in \mathbb{R}^3,$$

where G is the gravitational constant ($G = 6.67422 \cdot 10^{-11} m^3 / (\mathrm{kgs}^2)$) and where F is the density function. As is well known, the gravitational potential of the Earth corresponding to an integrable and bounded density function F, satisfies the Laplace equation in the outer space $\Delta V = 0$ and the Poisson equation in the interior space $\Delta V = -4\pi F$. The Newton integral (1) and its first derivatives are continuous everywhere on $\mathbb{R}^3$, i.e., $V \in C^{(1)}(\mathbb{R}^3)$. The second derivatives are analytic everywhere outside the real Earth surface, but they have a discontinuity when passing across the surface. For computational reasons, the spline interpolation of the Earth's gravitational potential is usually done in a spherical framework ([2], [3], [4]).

In this work, however, the intention is to propose a spline approach for the real Earth. We are interested in introducing an appropriate Hilbert space structure on the space of potentials in $\overline{\Sigma^{ext}}$, generated by the Newton integral. In order to accomplish this, in [1] it was shown that associating the density function to the class of harmonic functions in $L^2(\Sigma^{int})$ is closely related to the Hilbert space structure. Following the decomposition

$$L^2(\Sigma^{int}) = N(\mathcal{P}) \oplus Harm(\Sigma^{int}),$$

where $Harm(\Sigma^{int}) = \{F \in L^2(\Sigma^{int}) \mid \Delta F = 0\}$ is the space of all regular harmonic $L^2(\Sigma^{int})$ - distributions, $\mathcal{P}$ is the Newtonian potential operator

$$\mathcal{P}F = \int_{\Sigma^{int}} \frac{F(y)}{|\cdot - y|} dy \tag{2}$$

and $N(\mathcal{P})$ denotes it's nullspace, it was shown that a norm can be introduced in the space $\mathcal{H} = \mathcal{P}(L^2(\Sigma^{int}))$ of potentials in $\overline{\Sigma^{ext}}$ representing the images of the density functions from $L^2(\Sigma^{int})$ under the Newton operator $\mathcal{P}$, by restricting the operator to the closed subspace $Harm(\Sigma^{int})$. For every $P \in \mathcal{H}$,

$$\| P \|_{\mathcal{H}} = \| F \|_{L^2(\Sigma^{int})}, \tag{3}$$

where F is the unique harmonic density function $F \in Harm(\Sigma^{int})$ such that $\mathcal{P}F = P$. Moreover, a scalar product can be defined in $\mathcal{H}$ by $(\mathcal{P}F, \mathcal{P}G)_{\mathcal{H}} = (F, G)_{L^2(\Sigma^{int})}$, for $F, G \in Harm(\Sigma^{int})$. Moreover, $\mathcal{H}$ is a reproducing kernel Hilbert space. Considering the kernel function $k : \overline{\Sigma^{ext}} \times \Sigma^{int} \to \mathbb{R}$ given by $k(x, y) = \frac{1}{|x - y|}$, it is clear that for a fixed $x \in \overline{\Sigma^{ext}}$, $k(x, \cdot)$ is an element of $Harm(\Sigma^{int})$ (it is an element of $C^{(\infty)}(\Sigma^{int})$ and harmonic in Σ^{int}). Thus, from (2) and our previous considerations, it is clear from Hilbert space theory that at the point $x \in \overline{\Sigma^{ext}}$

we can represent a given potential $P \in \mathcal{H}$ as $P(x) = \mathcal{P}F(x) = (F, k(x,\cdot))_{L^2(\Sigma^{int})}$, for some $F \in Harm(\Sigma^{int})$. It is remarkable that not only for $x \in \Sigma^{ext}$, but also for all points $x \in \overline{\Sigma^{ext}}$ the functional $L_x(P) = P(x)$ is a bounded functional on $\mathcal{H}$. Indeed, from the representation $P(x) = (F, k(x,\cdot))_{L^2(\Sigma^{int})}$ and Cauchy--Schwarz inequality we get $| P(x) |^2 \leq \| F \|^2_{L^2(\Sigma^{int})} \| k(x,\cdot) \|^2_{L^2(\Sigma^{int})}$. Then from (3), it follows that

(4) $$| P(x) |^2 \leq C(x) \cdot \| P \|^2_{\mathcal{H}},$$

for every $P \in \mathcal{H}$, where $C(x) = \| k(x,\cdot) \|^2_{L^2(\Sigma^{int})}$ for a fixed $x \in \overline{\Sigma^{ext}}$ .Thus the necessary and sufficient condition for a Hilbert space to possess a reproducing kernel. The reproducing kernel is given by [1]

(5) $$\mathcal{K}(x,\cdot) = \int_{\Sigma^{int}} \frac{dz}{| x - z \| \cdot - z |}, x \in \overline{\Sigma^{ext}}.$$

Obviously, for a fixed $x \in \overline{\Sigma^{ext}}$, the reproducing kernel $\mathcal{K}(x,\cdot)$ is a Newtonian potential corresponding to the harmonic density function $\frac{1}{| x - \cdot |}$ from $L^2(\Sigma^{int})$. Moreover, for a fixed $x \in \overline{\Sigma^{ext}}$, the potential $\mathcal{K}(x,\cdot)$ is harmonic function in Σ^{ext} and a continuous function on Σ. Indeed, for a fixed $x \in \overline{\Sigma^{ext}}$, the density $\frac{1}{| x - \cdot |}$ is an element of $L^1(\Sigma^{int})$. This fact assures that $\mathcal{K}(x,\cdot)$ satisfies the Laplace equation in Σ^{ext}. Moreover, the potentials corresponding to densities in $L^2(\Sigma^{int})$ are elements in $C^{(0)}(\mathbb{R}^3)$.

As already mentioned this is an extraordinary fact, since it means that now in interpolation methods we will be able to use potentials of the same nature as the Earth's gravitational potential, instead of using outer harmonic expressions (as in the spherically harmonic case [3] of harmonic splines) harmonic down to the Runge sphere. The reproducing kernel is available in

integral form for any geophisically relevant geometry (like ellipsoid, geoid, actual Earth's surface).

3. SOLUTION TO DISCRETE EXTERIOR DIRICHLET BVP

Discrete Exterior Dirichlet Boundary Value Problem of gravitational potential determination (DEDP) in principle refers to the situation when we have n measurements $\alpha_1, \alpha_2, ..., \alpha_n$ of the scalar field of Earth's gravitational potential U at the points $x_1, x_2, ..., x_n$ on the Earth's surface. Denote

$\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}}) = \mathrm{Pot}(\Sigma^{ext}) \cap C^{(0)}(\overline{\Sigma^{ext}})$,

$\mathrm{Pot}(\Sigma^{ext}) = \{U \in C^{(2)}(\Sigma^{ext}) \mid \Delta U = 0, U \text{ regular at infinity}\}$.

In accordance to our considerations in the previous section, we consider the following discrete boundary value problem adequate in gravitational theory for a general geometry:

Let Σ be a regular surface. Let $\{x_1, ..., x_N\}$ be a discrete set of N points on Σ and let $\alpha_i = U(x_i), i = 1, ..., N$, be a given data set corresponding to a function $U \in C^{(0)}(\Sigma)$. Find an approximation $U_N \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ to the potential $U : \overline{\Sigma^{ext}} \to \mathbb{R}$, $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ such that $U_N(x_i) = U(x_i) = \alpha_i, i = 1, ..., N$.

We consider the Dirichlet functional. From (4), we easily get that for each $x \in \overline{\Sigma^{ext}}$, the linear functional D_x defined by $D_x : P \mapsto D_x P = P(x), \quad P \in \mathcal{H}$ is bounded on $\mathcal{H}$, i.e., $| D_x P | = | P(x) | \leq C(x, \Sigma) \| P \|_{\mathcal{H}}$, where $C(x, \Sigma) = \| k(x, \cdot) \|_{L^2(\Sigma^{int})}$.

Moreover, for each point $x \in \overline{\Sigma^{ext}}$ the function $y \mapsto \mathcal{K}(x, y), \quad y \in \overline{\Sigma^{ext}}$ is an element of $\mathcal{H}$ and for all $P \in \mathcal{H}$, due to reproducing properties of $\mathcal{K}(x, \cdot)$ we have $D_x P = P(x) = (P, \mathcal{K}(x, \cdot))_{\mathcal{H}}$.

In other words, Dirichlet functional is bounded on the space of potentials $\mathcal{H}$. Our aim is to find the smoothest $\mathcal{H}$- interpolant corresponding to data set $\alpha_1, \alpha_2, ..., \alpha_n$, where by `smoothest' we mean that the norm is minimized in $\mathcal{H}$. In other words, the problem is to find a function $S^U_{D_1, ..., D_N}$ in the set

$\mathcal{I}^U_{D_1,...,D_N} = \{P \in \mathcal{H} \mid D_i P = \alpha_i, \quad i = 1,...,N\}$, such that $\left\| S^U_{D_1,...,D_N} \right\|_{\mathcal{H}} = \inf_{P \in \mathcal{I}^U_{D_1,...,D_N}} \|P\|_{\mathcal{H}}$.

The corresponding representer of the functional D_i can be written as $D_i \mathcal{K}(\cdot,\cdot) = \mathcal{K}(x_i,\cdot)$. Then, for a given set $D_1, D_2, ..., D_N$, corresponding to the set $X_N = \{x_1, ..., x_N\}$ of points on Σ, we have the set of representers $\{D_1\mathcal{K}(\cdot,\cdot),...,D_N\mathcal{K}(\cdot,\cdot)\}$. The reproducing property then yields, for $i = 1,...,N$, and $P \in \mathcal{H}$, $D_i P = (D_i \mathcal{K}(\cdot,\cdot), P)_{\mathcal{H}}$. Having in mind that the reproducing kernel is given as a Newton integral (1), so are the representers of the functionals D_i, i.e., $D_i \mathcal{K}(\cdot,\cdot) = \int_{\Sigma^{int}} \frac{dz}{|x_i - z||\cdot - z|}$.

The interpolating $\mathcal{H}$-spline function is defined as follows

$$S(x) = \sum_{i=1}^{N} a_i D_i \mathcal{K}(\cdot, x) = \sum_{i=1}^{N} a_i \int_{\Sigma^{int}} \frac{dz}{|x_i - z||x - z|}, \quad x \in \overline{\Sigma^{ext}}.$$

Now the problem of determining the smoothest function in the set of all $\mathcal{H}$-interpolants is related to a system of linear equations which needs to be solved to obtain the spline coefficients. Indeed, the application of the linear functionals $D_1, D_2, ..., D_N$ to the $\mathcal{H}$-spline yields N linear equations in the coefficients $a_1^N, ..., a_N^N$, $\sum_{j=1}^{N} a_j^N D_i D_j \mathcal{K}(\cdot,\cdot) = D_i U, \quad i = 1,...,N$. The elements of the coefficients matrix $\left(D_i D_j \mathcal{K}(\cdot,\cdot)\right)_{i,j=1,...,N}$ are given by $D_i D_j \mathcal{K}(\cdot,\cdot) = \int_{\Sigma^{int}} \frac{1}{|x_i - z||x_j - z|} dz$. Since the coefficient matrix as Gram matrix of the N linearly independent functions is non-singular, the linear system is uniquely solvable. Together with the set of linear bounded functionals and the reproducing kernel Hilbert space $\mathcal{H}$, the coefficients $a_1^N, ..., a_N^N$ define the unique interpolating spline we are looking for. The interpolation problem $\| S^U_{D_1,...,D_N} \|_{\mathcal{H}} = \inf_{P \in \mathcal{I}^U_{D\_1,...,D_N}} \| P \|_{\mathcal{H}}$

is well-posed in the sense that its solution exists, is unique, and depends continuously on the data.

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Quasi Components in Dynamical Systems

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***Abstract**: We shall discuss the connection between quasi components and limit sets in a given dynamical system. Also a result will be presented which connects chain transitive components and quasi components for locally compact metric spaces.*

***Keywords**: quasi component, limit set, chain recurrence*

1. INTRODUCTION

The main aim of this paper is to present some results for quasi components of subsets in a given dynamical system. Recall that quasi component of a point x in a topological space (X,τ) is the intersection of all clopen subsets of X which contain the point x. Quasi components are closed subset of X. The quasi components of two distinct points of a topological space (X,τ) either coincide or are disjoint, so that all quasi components constitute a decomposition of the space X into pair wise disjoint closed subsets, which are called the quasi components of the space X. An alternative description of quasi components can be given based on the notion of functional separation. If A and B are subsets of a topological space (X,τ) we say that A and B are functionally separated in X if there exists a continuous function $f: X \to \{0,1\}$ such that $f(A)=0$ and $f(B)=1$. Now quasi component $Q(x)$ of a point x consists of all points y which cannot be functionally separated from x ([3]). Before proceeding further we will recall some elementary concepts for continuous dynamical systems and fix notations.

Definition**.** Let (X,d) be a given metric space. A flow in X is one parameter family of homeomorphisms $\{\varphi_t \mid t \in \mathbb{R}\}$, $\varphi_t : X \to X$ such that they satisfy the following two conditions:

$$\varphi_0 = 1_X, \varphi_t \circ \varphi_s = \varphi_{t+s}, \forall s,t \in \mathbb{R}$$

Let us fix some conventions. We shall always reserve the letter φ for flows, and denote, as customary, $\varphi_t(p) = \varphi(p,t)$ by $p \cdot t$. The phase space will be assumed to be locally compact and metrizable.

Definition**.** We say that a given set $M \subseteq X$ is invariant under the flow φ if $\varphi(M,t) \subseteq M$, $\forall t \in \mathbb{R}$. By replacing the set $\mathbb{R}$ with $\mathbb{R}^+$ or $\mathbb{R}^-$ we obtain the corresponding notions of positive and negative invariant set.

The trajectory of a point x is the set $\gamma(x) = \{xt \mid t \in \mathbb{R}\}$. By replacing the set $\mathbb{R}$ with $\mathbb{R}^+ \cup \{0\}$ or $\mathbb{R}^- \cup \{0\}$ we obtain the corresponding notions of positive and negative semi trajectory. We denote by $\gamma^+(x)$ and $\gamma^-(x)$ correspondingly. Many questions concerning flows involve their long term behavior or, intuitively speaking, the values $\varphi(p,\infty) = p \cdot \infty$

We introduce positive limit set of a given subset $M \subseteq X$ with the following: $\omega(M) = \{x \mid \exists x_n \in M, \exists t_n \to +\infty,\ \varphi(x_n,t_n) \to x\}$.

Analogous we define negative limit set $\alpha(M)$. A point x is positively Poisson stable if $x \in \omega(x)$. The definition of negatively Poisson stable point is analogous.

2. CHAIN TRANSITIVE COMPONENTS

We will now introduce the concept of chain recurrence and the notion of chain transitivity. Let $P(X) = \{f \mid f : X \to \mathbb{R}^+, f \text{ is continuous}\}$.

Definition. For $x, y \in X$ and $\varepsilon \in P(X), T > 0$ an (ε, T) chain from x to y is given by a natural number $n \in \mathbb{N}$, together with points $x_0 = x, x_1, ..., x_n = y \in X$ and times $T_0, T_1, ..., T_{n-1} \geq T$, such that $d(x_i T_i, x_{i+1}) < \varepsilon(x_i T_i)$, for $i = 0,1,...,n-1$.

Note that the number n of jumps is not bounded.

Definition. A subset $Y \subseteq X$ is chain transitive if for all $x, y \in Y$ and $\varepsilon \in P(X), T > 0$ there exists an (ε, T)- chain from x to y. A point $x \in X$ is chain recurrent if for all $\varepsilon \in P(X), T > 0$ there exists an (ε, T)-chain from x to x. The chain recurrent set $CR(\varphi)$ is the set of all chain recurrent points.

It is easily seen that this set is closed an invariant. The maximal chain transitive subsets of $CR(\varphi)$ are called chain transitive components.

Example: Consider the complete metric space S^1, the $1-$dimensional sphere, which we indentify here with $R/2\pi$. On S^1 the differential equation

$\dot{x} = \sin^2 x$ defines a dynamical system. In this case we have $CR(\varphi) = S^1$ which is easily proved.

3. QUASI COMPONENTS AND CHAIN TRANSITIVITY

We will need the following lemma:

Lemma1. Let $\omega(x) = \bigcup_{k=1}^{n} C_k$, where C_k are the connected components of the limit set $\omega(x)$. For arbitrary compact subset K such that $K \cap C_j \neq \varnothing$, $d(K, C_i) > 0$, $i \neq j$, there exists a quasi component Q of $\overline{\gamma(x)} \setminus (K \cap \gamma(x)) \cup \omega(x)$ such that $\omega(x) \subseteq Q$.

Proof: The case $n = 1$ is trivial. We assume that $n \geq 2$. Let us note first that the following holds true $\omega(x) \cap \gamma(x) = \varnothing$. Namely, if we assume the opposite than using the invariance of the limit set we conclude that $x \in \omega(x)$ which means that the point x is positively Poisson stable. Now using the fact that the limit set $\omega(x)$ is closed we deduce that $\overline{\gamma(x)} \subseteq \omega(x)$ and hence $\omega(x) = \overline{\gamma(x)}$. But this means that the limit set $\omega(x)$ is connected and $n = 1$. This is a contradiction.

We will use the equality $\overline{\gamma(x)} \setminus (K \cap \gamma(x)) \cup \omega(x) = \overline{\gamma(x)} \setminus (K \cap \gamma(x))$ in what follows. Notice that for every component C_k the following holds true $C_k \setminus K \cap \overline{\gamma(x)} \neq \varnothing$. Namely, If we assume the opposite then C_k should be compact as a closed subset of a compact set. But then if $\omega(x)$ is not compact, according to theorem from [1], none of its components are compact which is a contradiction. The other case means that $\omega(x)$ is compact, but then $n = 1$ from the same theorem from [1]. Again a contradiction.

Let us choose an arbitrary component C_i such that $d(K, C_i) > 0$. We will show that the components C_i, C_j cannot be functionally separated in $\overline{\gamma(x)} \setminus K \cap \gamma(x)$. Namely, let us assume that a continuous map exists $f : \overline{\gamma(x)} \setminus K \cap \gamma(x) \to \{0,1\}$ such that $f(C_i) = 0$ and $f(C_j) = 1$. We introduce the following sets $U = f^{-1}(0)$ and $V = f^{-1}(1)$. The first one is an open neighbourhood for C_i and the second one is an open neighbourhood for C_j, i.e. $C_i \subseteq U$, $C_j \subseteq V$. We will construct a sequence $x\tau_n$, $\tau_n \to \infty$ sufficiently close to C_i such that $x\tau_n \in U$, $\forall n \in N$. Namely, it suffices to

choose a point $z \in C_i$ and notice that a sequence $\tau_n \to \infty$ exists such that $x\tau_n \to z$ and for sufficiently large n, $x\tau_n \in U$ because of $d(K, C_i) > 0$. The positive semi-trajectory $\gamma^+(x\tau_n)$ will enter the set V because a point $w \in C_j \setminus K \cap \overline{\gamma(x)}$ exists and also a neighbourhood W of w exists disjoint from K and small enough such that $W \subseteq V$. Let us suppose that the positive semi-trajectory $\gamma^+(x\tau_n)$ after leaving U intersects K before entering V. Let $\theta_n = \sup\{t \geq \tau_n : x[\tau_n, t\ ) \subseteq U\}$. Then the segment $x[\tau_n, \theta_n)$ is contained in U, i.e. $x[\tau_n, \theta_n) \subseteq U$ and $x\theta_n \in K$. From the compactness of K we can assume that the sequence $x\theta_n \to p \in K$. But having in mind that $\theta_n \geq \tau_n \to \infty$ we deduce that $p \in \omega(x)$. Now because C_j is the only connected component of $\omega(x)$ with non empty intersection with K we conclude that $p \in C_j$. Let us choose arbitrary sequence $\delta_n \to 0,\ \delta_n > 0$. From the definition of the sequence θ_n the following holds true $x(\theta_n - \delta_n) \in U$. But $x(\theta_n - \delta_n) = x\theta_n(-\delta_n) \to p0 = p \in C_j$ and hence from the continuity of f we obtain $0 = \lim_{n\to\infty}(f(x(\theta_n - \delta_n)) = f(\lim_{n\to\infty} x(\theta_n - \delta_n)) = f(p) = 1$. This a contradiction. The second possibility is that the positive semi-trajectory $\gamma^+(x\tau_n)$ after leaving U does not intersects K. But every continuous map from connected set to the set $\{0,1\}$ is a constant map. Again from the connectedness of the set $\gamma^+(x\tau_n)$ we arrive at a contradiction. So C_i, C_j are in the same quasi component of $\overline{\gamma(x)} \setminus (K \cap \gamma(x))$ for every i and hence $\omega(x) = \bigcup_{k=1}^{n} C_k \subseteq Q$.

Theorem1. Let $CR(\varphi) = \bigcup_i \omega(x_i) \cup \alpha(x_i)$, for some family of points $\{x_i \mid i \in I\}$ which fulfil the following condition:

There exists $c > 0$ such that $r_1(x_i) \cap r_2(x_j) = \varnothing \Rightarrow d(\gamma(x_i), \gamma(x_j)) \geq c > 0$ for arbitrary $i, j \in I$, where $r_1, r_2 \in \{\alpha, \omega\}$. Furthermore let K_i be compact sets such that $K_i \cap C_j^r \neq \varnothing$, $d(K_i, C_m^r) > 0$, $\forall m \neq j$ and C_m^r are the connected components of $r(x_i)$. Moreover let K_i be compact

neighbourhoods of points from $K_i \cap C_j^r$. Then an arbitrary chain transitive component is contained in a quasi component of $\bigcup_i \overline{\gamma(x_i)} \setminus (\gamma(x_i) \cap K_i)$.

Proof: First let us note the following relation $CR(\varphi) = CR(\varphi_{CR(\varphi) \cup (\bigcup_I \overline{\gamma(x_i)})}$ so it is sufficient to work with the flow restrictive on $CR(\varphi) \cup (\bigcup_i \overline{\gamma(x_i)})$ ([2]). We will show that the limit set $\omega(x_i)$ is a subset of a chain transitive component. Let us choose arbitrary points $p, q \in \omega(x_i)$. From the local compactness of the phase space it is sufficient to work with constants $\varepsilon > 0$ instead of functions $\varepsilon : X \to \mathbb{R}^+$. Namely, there exist compact neighbourhoods for pT, q in which $\varepsilon : X \to \mathbb{R}^+$ reaches it's minimum and maximum values. So it is sufficient to work with constants under these values. Let us choose $(\varepsilon, T), T > 0$. From the choice of p and the invariance of $\omega(x_i)$, there exists $t_1 \geq T$ such that $d(x_i t_1, pT) < \varepsilon$. Similarly, there exists $t_2 \geq T$ such that $d((x_i t_1)t_2, q) < \varepsilon$. Now the sequence $\{p, x_i t_1, q; T, t_2\}$ is an $(\varepsilon, T)-$chain from p to q. Analogous, there exists an $(\varepsilon, T)-$chain from q to p. So p and q are contained in one chain transitive component. The conclusion follows. Similarly, $\alpha(x_i)$ is a subset of a chain transitive component. Now we will show that the chain transitive components contain those limit sets such that for any two of them $\{r_a(x_i), r_b(x_j)\}$, where $r_a, r_b \in \{\alpha, \omega\}$ there exists a finite sequence $r_1(x_{k_1}), r_2(x_{k_2}), ..., r_n(x_{k_n})$ where $r_i \in \{\alpha, \omega\}$ such that:

(1) $$r_a(x_i) \cap r_1(x_{k_1}) \neq \varnothing,\ r_l(x_{k_l}) \cap r_{l+1}(x_{k_{l+1}}) \neq \varnothing,\ \forall l = 1, ..., n-1,$$
$$r_n(x_{k_n}) \cap r_b(x_j) \neq \varnothing$$

Let us choose arbitrary points $p \in r_a(x_i)$, $q \in r_b(x_j)$. There exist

$$z_c \in r_c(x_{k_c}) \cap r_{c+1}(x_{k_{c+1}}),\ c = 1, 2, ..., n-1,\ z_0 \in r_a(x_i) \cap r_1(x_{k_1}),$$
$$z_n \in r_n(x_{k_n}) \cap r_b(x_j)$$

We choose (ε, T). Again using local compactness we assume that $\varepsilon > 0$ is a constant. Namely, we can take minimum of all the minimum values that are reached in the compact neighbourhoods of the points $pT, z_c T, z_c, q$. Now imitating the previous discussion there exists an (ε, T)- chain from p to z_0, from z_0 to z_1,..., from z_n to q. Joining them together provides us with an

(ε,T)-chain from p to q. As we mentioned in the beginning it is sufficient to work on the set $CR(\varphi)\cup\bigcup_i\overline{\gamma(x_i)}$. We shall prove that if $p\in r_1(x_i)$, $q\in r_2(x_j)$ and the condition (1) is not fulfilled for the pair $\{r_1(x_i),r_2(x_j)\}$ then the points p and q are not in the same chain transitive component. Let us choose $\varepsilon<c$ sufficiently small such that every ε-jump from p must be on the trajectory $\gamma(x_i)$ or on trajectory $\gamma(x_l)$ such that $r_1(x_i)\cap r(x_l)\neq\emptyset$ or on the limit sets $r(x_l),r_1(x_i)$. The possibility $r_1(x_i)$ will return us on the beginning. So let's discuss the possibility that the jump is on the trajectory $\gamma(x_i)$. But then every jump from this trajectory can be on a limit set with non empty intersection with $r_1(x_i)$ or on a trajectory which has a limit set with non empty intersection with $r_1(x_i)$. So we only need to see the ε-jump on $\gamma(x_l)$. But then every jump, again, can be only on a limit set with non empty intersection with $r(x_l)$ or on a trajectory which has a limit set with non empty intersection with $r(x_l)$. Finitely many times repeating these steps can only lead us to a limit set $r(x_g)$ such that the couple $\{r_1(x_i),r(x_g)\}$ satisfies the condition (1). This proves the claim. Now according to lemma 1, for arbitrary index i, the limit set $r(x_i)$ is contained in a quasi component of $\overline{\gamma(x_i)}\setminus(K_i\cap\gamma(x_i))$, and consequently in a quasi component of $\bigcup_i(\overline{\gamma(x_i)}\setminus(\gamma(x_i)\cap K_i))$. Now if the intersection of two limit sets is non empty then their union is in one quasi component. Hence every chain transitive component as union of limit sets which fulfil the condition (1) is in one quasi component of $\bigcup_i(\overline{\gamma(x_i)}\setminus(\gamma(x_i)\cap K_i))$.

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A Note on the Analytic Representations of Sequences in $L^2(\mathbb{R})$ Space

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Abstract. *In this paper we consider convergent sequences of functions in $L^2(\mathbb{R})$ space and their analytic representations, prove that the sequence of the analytic representations uniformly converges on compact subset of $\mathbb{C}\backslash\mathbb{R}$, and give additional result about the analytic representation of the boundary function.*

Keywords: *distributions, analytic representations, Fourier transform, inverse Fourier transform*

1. INTRODUCTION

The boundary values representation has been studied for a long time and from different points of view. One of the first results is that if $f \in \mathrm{L}^1(\mathbb{R})$ then the function

$$\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} >, \qquad z = x + iy, \;\; x \notin supp f$$

is the Cauchy representation of f, i.e.

$$\lim_{y \to 0+} \int_{-\infty}^{\infty} \left[\hat{f}(x+iy) - \hat{f}(x-iy)\right] \varphi(x) dx = \int_{-\infty}^{\infty} f(x)\varphi(x)dx,$$

for every $\varphi \in D$, where D is the Schwartz space of test functions.

Theorem 1 deals with convergent sequence of functions in $\mathrm{L}^2(\mathbb{R})$ space and similar result, but concerning functions in $\mathrm{L}^1(\mathbb{R})$**,** is part of Theorem 1 in [6]. Although the result is similar to that in [6], the proof completely differs, i.e. the proof in this paper uses some properties of the Fourier transform of a function, $\mathcal{F}(f)$ and the inverse Fourier transform, $\mathcal{F}^{-1}(f)$. The Parseval's formulas state that for the functions $f_1, f_2 \in L^2(\mathbb{R})$ or $f_1, f_2 \in L^1(\mathbb{R})$, it is true that

$$\int_{-\infty}^{\infty} \mathcal{F}(f_1, t)\, f_2(t) dt = \int_{-\infty}^{\infty} f_1(\omega) \mathcal{F}(f_2, \omega)\, d\omega$$

and

$$\int_{-\infty}^{\infty} \mathcal{F}^{-1}(f_1,t)\, f_2(t)dt = \int_{-\infty}^{\infty} f_1(\omega)\mathcal{F}^{-1}(f_2,\omega)\, d\omega.$$

In [2], it is proved the following result: For $g \in \mathrm{L}^2(\mathbb{R})$ and $f(t) = \mathcal{F}^{-1}(g,t)$, the Cauchy representation of f is

$$\hat{f}(z) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{0} g(\omega)e^{-i\omega z}d\omega\,, & for\ y > 0, \\ -\dfrac{1}{2\pi}\displaystyle\int_{0}^{\infty} g(\omega)e^{-i\omega z}d\omega\,, & for\ y < 0. \end{cases}$$

In the proof of Theorem 1 we use the following consequence of this result, i.e. the result:

$$\mathcal{F}^{-1}\big(H(t)e^{itz},\omega\big) = \frac{1}{2\pi i(\omega - z)}, \qquad for\ y > 0$$

and

$$\mathcal{F}^{-1}\big(H(-t)e^{itz},\omega\big) = \frac{-1}{2\pi i(\omega - z)}, \qquad for\ y < 0,$$

where $H(t)$ is the Heaviside function.

2. MAIN RESULTS

Theorem 1. Let $\{f_n(t)\}$ be a sequence of functions in $\mathrm{L}^2(\mathbb{R})$ that converges to $f(t)$ in $\mathrm{L}^2(\mathbb{R})$ as n tends to ∞ and let

$$\hat{f}_n(z) = \frac{1}{2\pi i} < f_n(t), \frac{1}{t - z} >, \qquad Imz \neq 0.$$

Then:

i) For every $n \in \mathbb{N}$, $\hat{f}_n(z)$ is an analytic representation of $f_n(t)$.

ii) The sequence $\{\hat{f}_n(z)\}$ uniformly converges on every compact subset of $\mathbb{C}\backslash\mathbb{R}$ to the function $\hat{f}(z)$, where

$$\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t - z} >, \qquad Imz \neq 0.$$

iii) The function $\hat{f}(z)$ is an analytic representation of the function $f(t)$.

Proof. i) Let $z = x + iy$ be a complex number such that $Imz \neq 0$. For any $\varphi \in D$ and $n \in \mathbb{N}$,

$$I = \int_{-\infty}^{\infty} \big[\hat{f}_n(x + iy) - \hat{f}_n(x - iy)\big]\varphi(x)dx =$$

$$\int_{-\infty}^{\infty} \frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty}\left[\frac{f_n(t)}{t - x - iy} - \frac{f_n(t)}{t - x + iy}\right]dt\right\}\varphi(x)dx.$$

Using the formulas

$$\frac{1}{2\pi i(t-x-iy)} = \mathcal{F}^{-1}\big(H(\omega)e^{i\omega(x+iy)}, t\big), \qquad for\ \ y>0$$

and

$$\frac{-1}{2\pi i(t-x-iy)} = \mathcal{F}^{-1}\big(H(-\omega)e^{i\omega(x+iy)}, t\big), \qquad for\ \ y<0$$

and the Parseval's formula, we get that

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_n(t)\mathcal{F}^{-1}\big(H(\omega)e^{i\omega(x+iy)}, t\big)dt + \int_{-\infty}^{\infty} f_n(t)\mathcal{F}^{-1}\big(H(-\omega)e^{i\omega(x-iy)}, t\big)dt\right]\varphi(x)dx =$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} H(\omega)e^{i\omega(x+iy)}\mathcal{F}^{-1}(f_n,\omega)d\omega + \int_{-\infty}^{\infty} H(-\omega)e^{i\omega(x-iy)}\mathcal{F}^{-1}(f_n,\omega)d\omega\right]\varphi(x)dx =$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} H(\omega)e^{i\omega x}e^{-\omega y}\mathcal{F}^{-1}(f_n,\omega)d\omega + \int_{-\infty}^{\infty} H(-\omega)e^{i\omega x}e^{\omega y}\mathcal{F}^{-1}(f_n,\omega)d\omega\right]\varphi(x)dx =$$

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} e^{i\omega x}\mathcal{F}^{-1}(f_n,\omega)[H(\omega)e^{-\omega y} + H(-\omega)e^{\omega y}]d\omega\right\}\varphi(x)dx.$$

Using Fubini's theorem, we get that

$$I = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\varphi(x)e^{i\omega x}dx\right]\mathcal{F}^{-1}(f_n,\omega)[H(\omega)e^{-\omega y} + H(-\omega)e^{\omega y}]d\omega =$$

$$\int_{-\infty}^{\infty}\mathcal{F}(\varphi,\omega)\mathcal{F}^{-1}(f_n,\omega)[H(\omega)e^{-\omega y} + H(-\omega)e^{\omega y}]d\omega.$$

By the Lebesgue dominated convergence theorem, we get that

$$\lim_{y\to 0+} I = \int_{-\infty}^{\infty}\mathcal{F}(\varphi,\omega)\mathcal{F}^{-1}(f_n,\omega)\,d\omega.$$

One more use of the Parseval's formulas gives that

$$\lim_{y\to 0+} I = \int_{-\infty}^{\infty} f_n(t)\mathcal{F}^{-1}\big(\mathcal{F}(\varphi,t)\big)\,dt = \int_{-\infty}^{\infty} f_n(t)\varphi(t)\,dt.$$

So, we proved that $\hat{f}_n(z)$ is an analytic representation of $f_n(t)$ for every $n \in \mathbb{N}$.

ii) Now let $z = x + iy$ be a complex number such that $Imz \neq 0$. Using the definitions of $\hat{f}_n(z)$ and $\hat{f}(z)$, we get that

$$\hat{f}_n(z) - \hat{f}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f_n(t)}{t-z}dt - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f_n(t)-f(t)}{t-z}dt.$$

Since $f_n, f \in L^2(\mathbb{R})$ for every $n \in \mathbb{N}$ and $\frac{1}{t-z} \in L^2(\mathbb{R})$, for $Imz \neq 0$, we may apply the Schwarz's inequality. So, the following inequality

$$\left|\hat{f}_n(z) - \hat{f}(z)\right| \leq \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}|f_n(t)-f(t)|^2dt\right]^{1/2}\left[\int_{-\infty}^{\infty}\frac{1}{|t-z|^2}dt\right]^{1/2}$$

holds.

By the assumption, the sequence $\{f_n(t)\}$ converges to the function $f(t)$ in $L^2(\mathbb{R})$ as n tends to infinity. So, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\left[\int_{-\infty}^{\infty}|f_n(t)-f(t)|^2dt\right]^{1/2} < \varepsilon.$$

We know that the quality

$$\left[\int_{-\infty}^{\infty}\frac{1}{|t-z|^2}dt\right]^{1/2}$$

is bounded.

Finally, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $z \in \mathbb{C}$, for which $Imz \neq 0$, the inequality

$$\left|\hat{f}_n(z) - \hat{f}(z)\right| < \varepsilon$$

holds, which proves ii).

The proof of iii) can be done in the same way as the proof of i), so we will not be repeating it.

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Valuation of Dependencies at a Given Intensity of Model Elements

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Abstract: *Some application models as geomechanical are characterized by large errors of main parameters. It is therefore necessary to evaluate the accuracy of the dependencies.*

In the paper a criterion for determining the local density of the model elements in to the graph of parameters is discussed.

Different cases of the method of valuation of dependencies are considered.

Keywords: *recoverability of image, local density of model elements, point processes*

1. INTRODUCTION

Let in a given model the relationship between the parameters is expressed by a continuous function F on values from $\mathrm{X} \subset \mathrm{R}^{\mathrm{m}_0}$ to $\mathrm{Y} \subset \mathrm{R}^{\mathrm{n}_0}$ $\mathrm{Y} = \mathrm{F}(\mathrm{X}) : \mathrm{X} \to \mathrm{Y}$. As a result, each pair of points $A_0, A \in \mathrm{X}$ is transformed respectively in $\mathrm{F}(A_0) = \mathrm{B}_0$, $\mathrm{F}(A) = \mathrm{B}$ and $\mathrm{B}_0, \mathrm{B} \in \mathrm{Y}$. To obtain an estimate of the model described analytically by F is sufficient to examine the transformation of the distance $\|AA_0\|$ in $\|BB_0\|$. Let be given segment $\overline{\mathrm{A}_0\mathrm{A}} = \left\{C : \overrightarrow{A_0C} = t\overrightarrow{A_0A}, t \in [0,1]\right\}$ where, $\overline{A_0A} \in \mathrm{X}$ and $S \in \overline{A_0A}$ $A = (x_1, x_2, ..., x_{m_0})$, $A_0 = (x_1^0, x_2^0, ..., x_{m_0}^0)$ and $\mathrm{S} = (s_1, s_2, ..., s_{m_0})$. Let the line does not cross the coordinate plane - i.e. coordinates of the line points are different from zero. Define the relative distance:. $\|AA_0\|_{\mathrm{S}} = \sqrt{\sum_{\mathrm{j}} \left(\frac{\mathrm{x_j} - \mathrm{x_j^0}}{\mathrm{s_j}} \right)^2}$ By analogy with the theory of the sensitivity of the computational tasks [5, 7] for evaluation of F can be used the ratio $\mu'_\varepsilon = \frac{\|\mathrm{BB}_0\|}{\|\mathrm{AA}_0\|_{\mathrm{S}}}$ where $S \in \overline{\mathrm{A}_0\mathrm{A}}$ and $\|A_0A\| \le \varepsilon$.

Let in the range $\mathrm{U} \subset \mathrm{R}$ we have defined parameter v. We say that this parameter is with a *permissible absolute error* ε if for each two values

$v_1, v_0 \subset U$ of the parameter is fulfilled $|v_1 - v_0| \le \varepsilon$ when these values are indiscernible on the subject of study - ie by technological or other application considerations both values are treated as indiscernible.

Let the range U does not contain the number zero. We say that v is *admissible relative error* ε if for each two values $v_1, v_0 \subset U$ of the parameter is fulfilled $\left|\frac{v_1 - v_0}{v_0}\right| \le \varepsilon$ when v_1 and v_0 are indiscernible on the subject of study.

Let $A = (x_1, x_2, \ldots, x_{m_0} \in X \subset R^{m_0})$. If ε_j is a permissible absolute error j - component of A, we call $\varepsilon = \|(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\|$ *admissible absolute error* of data from X. Similarly we define *permissible relative error*.

We say that the continuous dependence F is presented discreetly when a table $\overline{F}$ is selected from a finite number of values $T_1, T_2 \ldots, T_n$ of the argument and their respective functional values. In each partial region $U \subset X$ ($U \subset Y$) can be selected discrete grid of points $T_1, T_2 \ldots, T_n$ which we will refer to as *model elements*, such that each point $A \in X$ is indiscernible on the subject of study about some T_i. As in the case when $T_1, T_2 \ldots, T_n$ forming a rectangular grid, and will generally believe that model elements have spatial distribution of the Poisson point process.

Let $\varepsilon \ge 0$ and $\delta \ge 0$ are respectively permissible relative error for $X \subset R^{m_0}$ and absolute error for $Y \subset R^{n_0}$. We say that the discrete representation $\overline{F}$ of $F: X \to Y$ *can recover* if for every two recognizable on the subject of study points $B_0, B \in Y$, $\|B_0 B\| > \delta$ there are respective recognizable on the subject of study $A_0, A \in X$, such that $\|A_0 A\|_S > \varepsilon$ and it is met $F(A_0) = B_0$ and $F(A) = B$. Let $\varepsilon \ge 0$ and $\delta \ge 0$ are respectively permissible relative error and absolute error. Then it can be shown that if there are two recognizable on the subject of study $B_0, B \in Y$ such that it is performed there are appropriate $A_0, A \in X$ such is fulfilled.

(1) $$\frac{\|BB_0\|}{\|AA_0\|_S} \le \frac{\delta}{\varepsilon},$$

then the discrete representation $\overline{F}$ of F can recover.

In the article [1] are discussed various methods to restore an image from discrete representation.

2. MODEL ELEMENTS AS RANDOM POINT PROCESS

Let in a limited area $U \subset X \subset R^{m_0}$ (or $U \subset Y \subset R^{n_0}$) are given n points $T_1, T_2 \ldots, T_n$ and the indicator values $k = k(x_1, x_2, ..., x_{m_0})$ in these points – respectively $k_1, k_2, \ldots, k_n$. For discrete representation of the indicator k points $T_1, T_2 \ldots, T_n$ are model elements.

Let T_0 be any point on the field U. Model points $T_1, T_2 \ldots, T_n$ are numbered, so the their distances from point T_0 to be enforced $D_1 \le D_2 \le \le \ldots \le D_n$. By means of geometric probability [4] determines the distribution of distances to m-th point $D_m = \|T_0 T_m\|$ as $\frac{e^{-\lambda V}}{m!} d(\lambda V)^m$.

In [2] is obtained for the mathematical expectation $E(D_m)$

(2) $$q_m^{m_0} E(D_m) = E(D_1),$$

where $q_m^{m_0} = \frac{m_0}{m_0 + 1} \cdot \ldots \cdot \frac{m_0(m-1)}{m_0(m-1)+1}$, $E(D_1) = \frac{1}{\sqrt[m_0]{\lambda p_{m_0}}} r_{m_0}$ and $r_{m_0} = \int_0^\infty e^{-t^{m_0}} dt$.

For calculating the local density of the points on the model from (2) we obtained for $E(D_m)$

(3) $$\sqrt[m_0]{p_{m_0}} r_{m_0} q_m^{m_0} E(D_m) = \frac{1}{\sqrt[m_0]{\lambda}},$$

Consider the numbers $A_m = \sqrt[m_0]{p_{m_0}} r_{m_0} q_m^{m_0} D_m$ as estimates of random variable $\frac{1}{\sqrt[m_0]{\lambda}}$. Define local density L model points $T_1, T_2, \ldots, T_n$ around point T_0 by gender

(4) $$\frac{1}{\sqrt[m_0]{L}} = \sum_{m=1}^{n} \frac{A_m}{D_m} \Big/ \sum_{m=1}^{n} \frac{1}{D_m}.$$

Therefore

(5) $$L = d_n \left(\sum_{m=1}^{n} \frac{1}{D_m} \right)^{m_0}, \text{ where } d_n = 1 \Big/ \sum_{m=1}^{N} \sqrt[m_0]{p_{m_0}} r_{m_0} q_m^{m_0}.$$

Consider the set of model elements $T_1, T_2 \ldots, T_n$, each of them T_i associate with the polyhedron P_i, composed of elements of U, for which the point T_i is nearest. A random mosaic called Voronoi diagram is obtained [8].

In Poisson distribution of $T_1, T_2 \ldots, T_n$ with density λ we can determine the volume of a typical m_0 - dimensional polyhedron equal to $\frac{1}{\lambda}$. [8]. Accordingly $\frac{1}{\sqrt[m_0]{\lambda}}$ is a characteristic length of polyhedron by multitude $\{P_i\}$.

More generally in non-stationary point process of model elements we can use formula (4) to determine locally the amount of typical polyhedron. Let λ and μ are local (or total) densities of model elements, respectively, X and Y. These densities are compared relative distance in X and absolutely - in Y. Thus the term $\frac{\delta}{\varepsilon}$ can be replaced with $\frac{\sqrt[m_0]{\lambda}}{\sqrt[n_0]{\mu}}$.

3. PROPERTIES OF GRADIENT AND QUANTITATIVE VALUATION OF DEPENDENCES

Consider the length of the gradient $\|grad_{\ln}(\mathrm{f}(E))\| = \sqrt{\sum_{j=1}^{n}\left(\frac{\partial \mathrm{f}(\mathrm{E})}{\partial \ln x_j}\right)^2}$, expressed in semi-logarithmic derivative $\frac{\partial f(x)}{\partial \ln x} = x\frac{\partial f(x)}{\partial x}$.

In [3] are formulated and proven properties:

Property 1. For each $\varepsilon > 0$, there is such $\mathrm{E} \in \overline{\mathrm{A}_0\mathrm{A}}$ that

(6) 1. $$\frac{\|BB_0\|}{\|AA_0\|_E} \le \sqrt{\sum_j\left(\frac{\partial f(E)}{\partial lnx_j}\right)^2} = grad_{ln}(f(\mathrm{E})),$$

(7) 2. $$\|\mathrm{BB}_0\| \le \sqrt{\sum_j\left(\frac{\partial f(E)}{\partial lnx_j}\right)^2} \cdot \sqrt{\sum_j\left(\frac{x_j - x_j^0}{\mathrm{s}_j}\right)^2 \frac{\mathrm{s}_\mathrm{j}^2}{\mathrm{e}_\mathrm{j}^2}},$$

Property 2. Is fulfilled

(8) $$\frac{\|BB_0\|}{\|AA_0\|_S} = \sqrt{\sum_j \left(\frac{\partial f(\mathrm{A}_0)}{\partial\, lnx_j}\right)^2} + O(\|AA_0\|),$$

for each $\mathrm{S} \in \overline{\mathrm{A}_0\mathrm{A}}$, if $\overline{\mathrm{A}_0\mathrm{A}}$ not crosses a coordinate plane.

As a result, the condition of recoverability is written in the form

(9) $$\|\mathrm{grad}_{ln}\, f\| < \frac{\delta}{\varepsilon},$$

The disadvantages of such evaluation are introduced:

1. The evaluation $\|\mathrm{grad}_{\ln}(f)\|$ is not improvement but statistical probability for its gained is very small. In this sense, this assessment is increased;
2. In this type of evaluation by the gradient can be used maintly for qualitative conclusions;
3. There is'n an optimal solution of the problem because they do not account the randomness of the data.

In [3] is displayed evaluation of the dependencies, such as the above condition is fulfilled in confidence interval.

Final condition for recoverably is recorded in the form

(10) $$\mathrm{k} \cdot (1-\mathrm{p}) \cdot \|grad_{ln} f\| \cdot \le \frac{\delta}{\varepsilon},$$

where $\mathrm{k} = cos\,\varphi_0$, $\varphi_0 \in \left[0, \frac{\pi}{2}\right]$ и $p = \frac{V(\varphi_0)}{V} = \frac{\int_0^{\varphi_0} \sin^{m-2}\varphi\, d\varphi}{\int_0^{\frac{\pi}{2}} \sin^{m-2}\varphi\, d\varphi}$.

4. CONCLUSIONS

Method for valuation of the dependencies used as a criterion density of the model elements taken into account the values on the graph of dependence.

In comparison with the used safety factor, the proposed method for evaluating dependencies work with much more information: account required accuracy of the result and the gradient of the functional dependence and allows the determination of the parameters of model elements.

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Study of Random Mosaics Applicable in Crack Patterns Designing

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Abstract: The article describes some models of stochastic networks - the *isotropic Poisson line process and more complicated models.*

Basic relations for mean values are presented. A method for determination of symmetry relation and obtain appropriate centerline in vector segments (crack patterns) are discussed.

Keywords: *random mosaics, isotropic Poisson line process, crack pattern, recognition of line in raster image.*

1. INTRODUCTION

Random tessellations, also known as random mosaics or stochastic networks, are random partitions of the plane into disjoint regions. Mosaics arise naturally in many contexts. Examples include tilings, crystals, cellular structures, land use maps, galaxies, communication networks, crack patterns, foams, and so on. The set $\{P\}$ of limite, opened and convex polyhedrons is called random mosaic if:

1. $P_1 \cap P_2 = \varnothing$ for $P_1, P_2 \in \{P\}$, $P_1 \neq P_2$; 2. $\bigcup_{P \in \{P\}} \overline{P} = R^n$;

3. Each restricted set $M \subset R^n$ is crossed by a finite number of elements from $\{P\}$ [5].

Example of random mosaic is model of cracked rock [8].

If $\{P\}$ is a stationary random mosaic in R^3 then A_0 - set of the vertices; A_1 - set of the edges middles; A_2 - set of the centers of the polyhedron walls and A_3 - set of the centers of polyhedrons are stationary point processes. If $\lambda_i, i = 0,1,2,3$ is a the respective density of A_i then

$$\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3 = 0 \quad (1)$$

And $1/\lambda_3$ is the volume of a typical polyhedron of random mosaic. Other equations for the mean values are given in [5].

2. RANDOM MOSAIC AND POISSON PROCESSES

Generally Voronoi mosaic is defined by:

1. Let by given the stationary Poisson point process $\{x_1, x_2, \cdots\} \subset R^n$ with intensity λ;

2. For every element $x_{i_0} \in \{x_1, x_2, \cdots\}$ is compared the set

$P_{i_0} = \{y : \|x_{i_0} - y\| \le \|x_j - y\|, j \ne i_0\}$.

For this type of mosaic λ_0 is the density vertices; λ_1 - density of edges and $\lambda_2 = \lambda$ - density of the polygons. Met are the equations $\lambda_0 = 2\lambda$, $\lambda_1 = 3\lambda$. The average number of incoming edges at the same vertex is $s_0 = 3$. The average number of vertices of the polygon is typical. The average length of a typical edge - $k = \frac{2}{3}\sqrt{\lambda}$. The area of a typical polygon is $\frac{1}{\lambda}$.

Mosaic of lines in R^2 is called breaking the plane by a stationary process of lines. Any straight from R^2 comparing parameters $\theta \in [0, \pi)$ and $r \in R$ so the equation of line in Cartesian coordinates to be

(2) $$x\cos\theta + y\sin\theta = r$$

Defined thus radom set Ψ accidentally multiple is defined by:

1. Defined random point process Φ in $[0, \pi) \times R$ with measure where $\mu = \chi \times \nu$ extreme χ is finite measure in $[0, \pi)$ and ν - a measure of Lebeg in R;

2. $\Psi = \{g(\theta, r) : (\theta, r) \in \Phi\}$.

In such introduced measure Φ is a Poisson point process and therefore Ψ also is a Poisson point process - stationary and isotropic.

By analogy, we can define a stationary Poisson process $\{\alpha_i\}$ of plains by the equation

(3) $$ax + by + cz = r,$$

where $e_\alpha = (a, b, c)$ is a single normal vector of the plane.

In order stereomerically presentation of the plains of $\{\alpha_i\}$ are introduced parameters $\psi = \angle\left(\vec{Oz}, e_\alpha\right)$ - *dip angle* of the plane and $\beta = \angle\left(\vec{Oy}, f_1^\alpha\right)$ where f_1^α is directed projection of the reference line of the plane. The angle is

called *dip direction angle* of α. Then the coordinates of the normal vector e_α are

(4) $a = \sin\psi \sin\beta$, $b = \sin\psi \cos\beta$ и $c = \cos\psi$.

At each plane α are compared parameters $\psi \in [0,\pi)$, $\beta \in [0,2\pi)$ and $r \in [0,+\infty)$.

3. STEREOLOGY AND RECOGNITION

By means of stereological establish the mean - values of essential parameters of three-dimensional mosaic given plane sections. Object of study of this material is the presentation of a mathematical model to determine the parameters of the fracture systems in the rock. Output data are fracture patterns that are obtained graphically from the walls of the mine workings. The stereological task is achieved by methods of computer recognition of sections of the a raster image.

3.1. Determination of the axis line in a bitmap image

Work of Blum [1], Brady [2] and Brooks [3] are the initial development on axial presentation of the form of a raster-objected objects in the computer. The method presented here to determine the axis line of the bitmap is applied in [6]. A common variant of this method is considered by Friedberg [7].

The set $\mathrm{M} = \{(\mathrm{x_i}, \mathrm{y_i})\}$ consists of raster elements. Let point U is the symmetry axis - the center of gravity of M where $\bar{x}_n = \frac{\sum \mathrm{x_i}}{\mathrm{n}}$, $\bar{y}_n = \frac{\sum \mathrm{y_i}}{\mathrm{n}}$ and n is the number of elements M.

Let be $\Lambda = \mathrm{m_{11}x^2} + 2\,\mathrm{m_{12}x\,y} + \mathrm{m_{22}y^2}$, is a quadratic form, where $\mathrm{m_{11}} = Var(x) = \sum(\mathrm{x_i} - \bar{x}_n)^2$, $\mathrm{m_{12}} = Cov(x,y) = \sum(\mathrm{x_i} - \bar{x}_n)(\mathrm{y_i} - \bar{y}_n)$ and $\mathrm{m_{22}} = Var(y) = \sum(\mathrm{y_i} - \bar{y}_n)^2$. Let the symmetrical axis is line s with a normal equation $x\cos\theta + ysun\theta = r$.

As a result of rotation $\mathrm{Oxy} \to \mathrm{Us}\,\ell$ with an angle $\varphi = \pi/2 - \theta$, quadratic form is converted into $\Lambda' = \mathrm{n_{11}s^2} + 2\,\mathrm{n_{12}s}\,\ell + \mathrm{n_{22}}\,\ell^2$.

Then

$$\Lambda'\begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}\Lambda = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}, \text{ where } \Lambda_\theta = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}.$$

Is obtained $n_{12} = \frac{1}{2}\left(-m_{11}\sin 2\theta + 2m_{12}\cos 2\theta + m_{22}\sin 2\theta\right) = 0$. $\Rightarrow$

$$\theta = \frac{1}{2}\operatorname{arctg}\frac{2m_{12}}{m_{11}-m_{22}} = \frac{1}{2}\operatorname{arctg}\frac{2Cov(x,y)}{Var(x)-Var(y)}, \quad (5)$$

3.2. Recognition and analytical presentation a crack pattern

3.2.1. Presentation of segment in (θ, r, t) spaceis

We will use the display of the segments from Oxy described by the normal equation $x\cos\theta + y\sin\theta = r$ in (θ, r) space - Hough transformation.

On fig. 1 is a section in (θ, r) space. Γ_1^θ and Γ_2^θ ate sets of points representing two opposite contour in length of the segment. In (θ, R) the section can be separated from other segments.

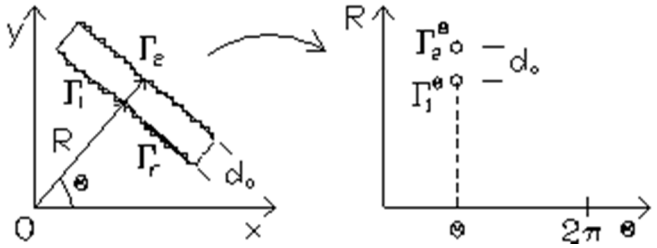

Fig. 1: Display of segment in (θ, r) space.

Use an addition of method to recognize lines with (θ, R) space [4]. Consider space (θ, r, t) that we present the segments, as a system of parametric equations

$$\begin{aligned} x &= r\cos\theta - t\sin\theta \\ y &= r\sin\theta + t\cos\theta \end{aligned} \quad (6)$$

With a natural parameter $t \in [a_t, b_t]$. Then in (θ, r, t) space the image of the segment is represented by parameters r, θ, a_t and b_t (Fig. 2).

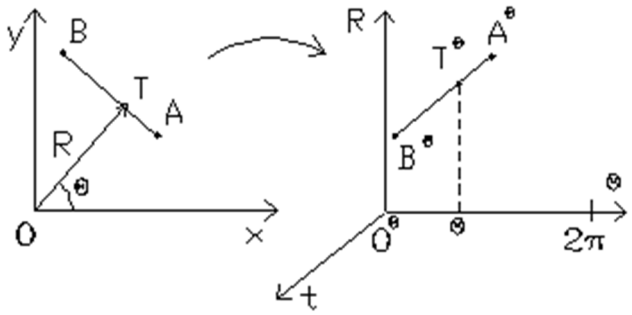

Fig. 2: Display a segment in (θ, r, t) space

3.2.2. Receive the parameters r, θ, a_t and b_t of a segment

Let a segment is represented by a set of raster pixels $M = \{(x_i, y_i)\}$.

You need to define the parameters θ, r, and $t_i \quad i = 1, 2, \cdots, n$,that meet the stochastic equations

$$\begin{aligned} \mathrm{x}_i &= r\cos\theta - \mathrm{t}_i\sin\theta \\ \mathrm{y}_i &= r\sin\theta + \mathrm{t}_i\cos\theta \end{aligned}. \tag{7}$$

The solution can be achieved by the method of least squares. Here we will apply a more rational method leading to the same result.

System (7) can be represented by matrix equation $\begin{pmatrix} x_i \\ y_i \end{pmatrix} = (r, t_i)\Delta_\theta$. After multiplying the right to equality with Δ'_θ and transposition is obtained $\begin{pmatrix} r \\ t_i \end{pmatrix} = \Delta_\theta \begin{pmatrix} x_i \\ y_i \end{pmatrix}$. Is obtained a system of equations

$$\begin{aligned} r &= \mathrm{x}_i\cos\theta + \mathrm{y}_i\sin\theta \\ t_i &= -\mathrm{x}_i\sin\theta + \mathrm{y}_i\cos\theta \end{aligned}. \tag{8}$$

$\Rightarrow t_i = -\mathrm{x}_i\sin\theta + \mathrm{y}_i\cos\theta$ for $i = 1, 2, \cdots, n$. From $r = \mathrm{x}_i\cos\theta + \mathrm{y}_i\sin\theta$ It follows, that if is given the parameter θ, then r can be accessed by mathematical expectation of the random variable $r = \bar{\mathrm{x}}_n\cos\theta + \bar{\mathrm{y}}_n\sin\theta$, where $\bar{x}_n = \frac{1}{n}\sum x_i$, и $\bar{y}_n = \frac{1}{n}\sum y_i$.

Angle θ accessed by formula (5). After determining of $t_i \quad i = 1, 2, \cdots, n$ we obtain the parameters $a_t = \min\limits_i x_i$ and $b_t = \max\limits_i x_i$.

4. STUDY DADA FOR CRACKS PLANES

From bitmap of mine workings wall identify fracture patterns. They represent sets of lines similar in shape to parallel sections. By means of computer graphics outlines of these sets. For each image of the crack parameters (θ, r, a_t, b_t) are obtained and crack vector $\vec{j} = \frac{\vec{P_1P_2}}{\|P_1P_2\|}$, where $P_1(\theta, r, a_t)$, $P_2(\theta, r, b_t)$.

Let crack vectors $\vec{j}_1$ and $\vec{j}_2$ are non-parallel and are of the same system. The normal vector of the system of the crack planes is $e_\alpha = \vec{j}_1 \times \vec{j}_2$. From the equations (4) we obtain the dip angle and dip direction angle of the system of cracks.

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Multivariate Integration in the Hilbert Space $H_{W(b),\alpha,\beta,\gamma}$

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Abstract: *We introduce a reproducing kernel Hilbert space $H_{W(b),\alpha,\beta,\gamma}$ and present a formula for the worst-case error of the integration in this space. The asymptotic behavior of the worst-case error is obtained.*

Key words: *reproducing kernel Hilbert space, quasi-Monte Carlo method, worst-case error*

1. INTRODUCTION

The Walsh functions in base b will be the main tool of our study. So, let $b \geq 2$ be an arbitrary integer. Following Chrestenson [3] we will recall the definition of the Walsh functions. For a non-negative integer k and a real x with the b −adic representation $k = k_0 + k_1 b + \ldots + k_\nu b^\nu$ and $x = \frac{x_0}{b} + \frac{x_1}{b^2} + \cdots$ where $k_i, x_i \in \{0, 1, \ldots, b-1\}$, $k_\nu \neq 0$, and infinitely many $x_i \neq$ b $- 1$, the corresponding Walsh function ${}_b wal_k : \longrightarrow \mathbb{C}$ is defined by

$$ {}_b wal_k(x) = e^{\frac{2\pi i}{b}(x_0 k_0 + \cdots + x_\nu k_\nu)}.$$

The set $W(b) = \{ {}_b wal_k(x) : k = 0, 1, \ldots ; \ x \in [0,1)^s\}$ is called Walsh functions in base b. The set $W(2)$ is the original Walsh [7] functional system.

Let $\mathbb{N}_0$ denote the set of non-negative integers.

For a dimension $s \geq 2$ and vectors $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we define ${}_b wal_{\boldsymbol{k}} : [0,1)^s \longrightarrow \mathbb{C}$ as ${}_b wal_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} {}_b wal_{k_j}(x_j)$. For any dimension $s \geq 1$ the system $\{ {}_b wal_{\boldsymbol{k}}(\boldsymbol{x}) : \boldsymbol{k} \in \mathbb{N}_0^s, \ \boldsymbol{x} \in [0,1)^s \}$ is a complete orthonormal functional system in $L_2([0,1)^s)$.

Following Niederreiter [4], [5] we will recall the theoretical bases and the constructive principle of the so-called (t, m, s) − net in base $b \geq 2$. This class of sequences are with well distribution of their points in $[0,1)^s$ and they are very intensively used in the practical application of the quasi-Monte Carlo integration. So, we give the definition of a (t, m, s) − net.

Definition 1 Let $0 \leq t \leq m$ be given integers. A point set P consisting of b^m points in $[0,1)^s$ forms a (t, m, s) − net in base b, if every subinterval of the

form $\prod_{j=1}^{s}\left[\frac{a_j}{b^{d_j}}, \frac{a_j+1}{b^{d_j}}\right)$ of $[0,1)^s$, where for $1 \le j \le s$ $d_j \ge 0$ and $0 \le a_j < b^{d_j}$ are integers, with a volume b^{t-m}, contains exactly b^t points of P.

Let $b \ge 2$ be a given base. For $1 \le j \le s$ let C_j be a given $m \times m$ matrices over $\mathbb{Z}_b = \{0, 1, \dots, b-1\}$. Let an arbitrary integer n, such that $0 \le n \le b^m - 1$, have the b −adic representation $n = n_0 + n_0 b + \cdots + n_{m-1} b^{m-1}$. We identify n with the vector $\boldsymbol{n} = (n_0, n_1, \dots, n_{m-1})^T$. For $1 \le j \le s$ let us multiply the matric C_j by $\boldsymbol{n}$, so let

$$C_j . \boldsymbol{n} = \left(x_{j,n,1}, x_{j,n,2}, \dots, x_{j,n,m}\right)^T.$$

We define the points $x_{j,n} = \frac{x_{j,n,1}}{b} + \frac{x_{j,n,2}}{b^2} + \dots + \frac{x_{j,n,m}}{b^m}$ *and the vector* $\boldsymbol{x}_n = \left(x_{1,n}, x_{2,n}, \dots, x_{s,n}\right)$.

If for some integer t, $0 \le t \le m$, the point net $P_{b^m} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{b^m-1}\}$ is a (t, m, s) − net, then it is called a digital (t, m, s) − net in base b.

Now, we should introduce some notations and a special type of coefficients, which will play a crucial role in our research. Following Baldeaux, Dick, Leobacher, Nuyens and Pillichshammer [2] we present the following details: Let $k \ge 0$ be an arbitrary integer. Let the symbol $\#k$ mean the number of non-zero digits in the b −adic representation of k of the form $k = \sum_{i=1}^{\#k} k_{g_i} b^{g_i}$, where for $i = 1, 2, \dots \#k$, $k_{g_i} \in \{1, 2, \dots, b-1\}$ and $g_1 > g_2 > \cdots > g_{\#k} \ge 0$. Let us assume that $\#0 = 0$.

According to [2] for an arbitrary integer $k \ge 0$ we define the coefficients

$$\rho(\alpha; \beta; \gamma; b; k) = \begin{cases} 1, & if\ k = 0, \\ \gamma b^{-\alpha \sum_{i=0}^{\min(\#k, \beta)} g_i}, & if\ k \ne 0. \end{cases}$$

Now, we will present the multidimensional version of the coefficients. For a vector **k** = $(k_1, \dots, s_s) \in \mathbb{N}_0^s$ and a vector **γ** = $(\gamma_1, \dots, \gamma_s)$ of weights we define

$$R(\alpha; \beta; \gamma; \boldsymbol{k}) = \prod_{j=1}^{s} \rho(\alpha; \beta; \gamma_j; b; k_j). \quad (1)$$

2. AN INTEGRATION IN THE HILBERT SPACE $H_{W(b),\alpha,\beta,\gamma}$

Following Aronszajn [1], we will recall the concept of reproducing kernels for Hilbert spaces. Let F be a class of functions defined on E, forming a Hilbert space. The function $K(x, y)$ of $x, y \in E$ is called a reproducing kernel for F if the following properties hold:

For every $y \in E$ $K(x, y)$, as a function of x, belongs to F;

(Reproducing property) for every $y \in E$ and every $f \in F$ $f(y) = < f(x), K(x, y) >_x$ and the subscript x indicates that the inner product applies to function of x.

Let $H_s(K)$ be a Hilbert space with reproducing kernel $K(\boldsymbol{x},\boldsymbol{y})$: $[0,1)^{2s} \longrightarrow \mathbb{R}$ and a norm $\|.\|_{H_s(K)}$. We are interested in approximating the multivariate integral

$$I_s(f) = \int_{[0,1]^s} f(\boldsymbol{x})d\boldsymbol{x}, \quad f \in H_{s(K)}.$$

Let $N \geq 1$ be an arbitrary fixed integer. We will approximate the integral $I_s(f)$ through quasi-Monte Carlo algorithm with equal quadrature weights

$$Q_s(f;P_N) = \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n),$$

where $P_N = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ is a deterministic sample point net, composed of N points in $[0,1)^s$.

The *worst-case error* of the integration in the space $H_s(K)$ by using the quasi-Monte Carlo algorithm $Q_s(f;P_N)$ is defined as

$$e(H_s(K);P_N) = \sup_{f\in H_s(K),\|f\|_{H_s(K)\leq 1}} |I_s(f) - Q_s(f;P_N)|.$$

Sloan and Woźniakowski [6] propose to arrange the arguments $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots \boldsymbol{x}_s$, of the integrands in such a way that $\boldsymbol{x}_1$ is the most important coordinate, $\boldsymbol{x}_2$ is the next one, and so on. This is realized by associating nonincreasing positive weights $\gamma_1, \gamma_2, \dots, \gamma_s$ to the successive coordinate direction, so we have a vector of weights $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_s > 0$.

We will define the function

$$K(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} R(\alpha;\beta;\gamma;\boldsymbol{k}) \; {}_b wal_{\boldsymbol{k}}(\mathbf{x}) \overline{{}_b wal_{\boldsymbol{k}}(\mathbf{y})}, \; \boldsymbol{x},\boldsymbol{y} \in [0,1)^s,$$

where the coefficient $R(\alpha;\beta;\gamma;\boldsymbol{k})$ is defined by the equality (1). We can prove that the function $K(\boldsymbol{x},\boldsymbol{y})$ is a reproducing kernel, which generates the functional space $H_{W(b),\alpha,\beta,\gamma}$.

For two function $f, g \in H_{W(b),\alpha,\beta,\gamma}$ we define the inner product by the equality

$$< f, g >_{H_{W(b),\alpha,\beta,\gamma}} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} R^{-1}(\alpha;\beta;\gamma;b;\boldsymbol{k}) \hat{f}_{H_{W(b),\alpha,\beta,\gamma}}(\boldsymbol{k}) \overline{\hat{g}_{H_{W(b),\alpha,\beta,\gamma}}(\boldsymbol{k})},$$

where $\hat{f}_{H_{W(b),\alpha,\beta,\gamma}}$ means the Fourier coefficient of the function f.

The norm $\|.\|_{H_{W(b),\alpha,\beta,\gamma}}$ is defined as

$$\|f\|_{H_{W(b),\alpha,\beta,\gamma}} = \sqrt{< f, f >_{H_{W(b),\alpha,\beta,\gamma}}}.$$

Then the space $H_{W(b),\alpha,\beta,\gamma}$ is defined as

$$H_{W(b),\alpha,\beta,\gamma} = \left\{f: \|f\|_{H_{W(b),\alpha,\beta,\gamma}} < \infty\right\}.$$

Theorem 1 gives a formula for the the worst-case error of the integration in this space through the usage of an arbitrary net P_N.

Theorem 1 Let $\alpha > 1$ be a given real, $\beta \geq 1$ be a fixed integer and γ= $(\gamma_1, \gamma_2, \dots, \gamma_s)$ be an arbitrary vector of weights. Let $P_N = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ be an arbitrary net of N points in $[0,1)^s$. Then the worst-case error of the integration in the Hilbert space $H_{W(b),\alpha,\beta,\gamma}$ by using the net P_N satisfies the equality

$$e^2\left(H_{W(b),\alpha,\beta,\gamma}; P_N\right) = -1 + \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} R(\alpha;\beta;\gamma;\boldsymbol{k}) \quad {}_b wal_{\boldsymbol{k}}(\boldsymbol{x_n})\overline{{}_b wal_{\boldsymbol{k}}(\boldsymbol{x_m})},$$

where the coefficient $R(\alpha;\beta;\gamma;\boldsymbol{k})$ is defined by the equality (1).

For an arbitrary integer $k \geq 0$ with the $b-$adic representation $k = \sum_{i=0}^{\infty} k_i b^i$ and a fixed integer m we define the matrix $tr_m(k)$ as

$$tr_m(k) = (k_0, k_1, \dots, k_{m-1})^T.$$

Theorem 2 Let $P_{b^m} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{b^m-1}\}$ be an arbitrary $(t,m,s)-$ net over $\mathbb{Z}_b$ composed of b^m points in $[0,1)^s$. Let P_{b^m} be generated by the matrices $C_1, C_2, \dots, C_s$ and let define the set

$$\chi = \left\{\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s : C_1^T\, tr_m(\boldsymbol{k}_1) + \cdots + C_s^T\, tr_m(\boldsymbol{k}_s) = \boldsymbol{0}\right\}.$$

Then the worst-case error of the integration in the Hilbert space $H_{W(b),\alpha,\beta,\gamma}$ by using the net P_{b^m} satisfies the equality

$$e^2\left(H_{W(b),\alpha,\beta,\gamma}; P_{b^m}\right) = \sum_{\boldsymbol{k}\in\chi/\{\boldsymbol{0}\}} R(\alpha;\beta;\gamma;\boldsymbol{k}).$$

The following theorems hold:

Let $M_{b,m}$ be the set of $m \times m$ matrices over $\mathbb{Z}_b$.

Theorem 3 For an arbitrary real λ, $0 < \lambda \leq 1$ there is a set of matrices $C_1', C_2', \dots, C_s' \in M_{b,m}$, such that

$$e^{2\lambda}\left(H_{W(b),\alpha,\beta,\gamma}; P_{b^m}\right) \leq -1 + \prod_{j=1}^{s}\sum_{l_{j=0}}^{\infty} \rho^{\lambda}\left(\alpha;\beta;\gamma; b^m l_j\right) + \frac{1}{b^m}\prod_{j=1}^{s}\sum_{l_{j=0}}^{\infty}\sum_{k_j=0}^{b^m-1} \rho^{\lambda}\left(\alpha;\beta;\gamma; b^m l_j + k^*\right).$$

Theorem 4 Let $\alpha > 1$ be an arbitrary real, $\beta = 1$ and γ= $(\gamma_1, \gamma_2, \dots, \gamma_s)$ be an arbitrary vector of weights.

(i)(**upper bound**) Let $\frac{1}{\alpha} < \lambda \leq 1$ be an arbitrary number. Then there exists a digital $(t,m,s)-$net P_{b^m} over $\mathbb{Z}_b$, such that the worst-case error of

the integration in the Hilbert space $H_{W(b),\alpha,\beta,\gamma}$ by using the net P_{b^m} satisfies the equality

$$e^{2\lambda}\left(H_{W(b),\alpha,\beta,\gamma}; P_{b^m}\right) \leq -1 + \prod_{j=1}^{s}\left[1 + \frac{\gamma_j^{\lambda}(b-1)b^{\alpha\lambda}}{b^{\alpha\lambda}-b} b^{-\alpha\lambda m}\right] + \frac{1}{b^m}\prod_{j=1}^{s}\left[1 + \frac{\gamma_j^{\lambda}(b-1)b^{\alpha\lambda}}{b^{\alpha\lambda}-b}\right].$$

(ii) (**asymptotic behavior**)

If $1 < \alpha < 2$ *,then* $e\left(H_{(b),\alpha,\beta.\gamma}; P_{b^m}\right) = \mathcal{O}\left(b^{-\frac{m}{2}}\right)$.

If $\alpha \geq 2$ *and* $\alpha = 2\alpha_1$ $(\alpha_1 \geq 1)$*, then* $e\left(H_{W(b),\alpha,\beta,\gamma}; P_{b^m}\right) = \mathcal{O}(b^{-\alpha_1 m})$.

Theorem 5 Let $\alpha > 1$ be an arbitrary real, $\beta = 2$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ be an arbitrary vector of weights.

(i)(**upper bound**) Let $\frac{1}{2\alpha} < \lambda \leq 1$ be an arbitrary number. Then there exists a digital (t, m, s) −net P_{b^m} over $\mathbb{Z}_b$, such that the worst-case error of the integration in the Hilbert space $H_{W(b),\alpha,\beta,\gamma}$ by using the net P_{b^m} satisfies the equality

$$e^{2\lambda}\left(H_{W(b),\alpha,\beta,\gamma}; P_{b^m}\right) \leq -1 + \prod_{j=1}^{s}\left[1 + \frac{\gamma_j^{\lambda}(b-1)b^{\alpha\lambda}}{b^{\alpha\lambda}-b} b^{-\alpha\lambda m} + \frac{\gamma_j^{\lambda}(b-1)^2 b^{\alpha\lambda}}{(b^{\alpha\lambda}-1)(b^{2\alpha\lambda}-b)} b^{-2\alpha\lambda m}\right] + \frac{1}{b^m}\prod_{j=1}^{s}\left[1 + \frac{\gamma_j^{\lambda}(b-1)(b^{\alpha\lambda}+b)b^{\alpha\lambda}}{b^{2\alpha\lambda}-b}\right]$$

(ii) (**asymptotic behavior**)

If $1 < \alpha < 2$ *,then* $e\left(H_{W(b),\alpha,\beta,\gamma}; P_{b^m}\right) = \mathcal{O}\left(b^{-\frac{m}{2}}\right)$.

If $\alpha \geq 2$ *and* $\alpha = 2\alpha_1$ $(\alpha_1 \geq 1)$*, then* $e\left(H_{W(b),\alpha,\beta,\gamma}; P_{b^m}\right) = \mathcal{O}(b^{-\alpha_1 m})$.

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On a Mathematical Model Describing the Adaptive Immune Response to Viral Infection

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***Abstract**: A mathematical model of adaptive immune response to viral infection is formulated by six nonlinear ordinary differential equations. The model describes the interactions between virus, uninfected cells, infected cells, the adaptive immune response represented by cytotoxic T lymphocytes (CTL) and the antibodies and damaged part of the target organ. Theorems of existence, uniqueness and non-negativity of solution are proven. Primary and secondary immune responses against viral infection are investigated by numerical simulations, using Matlab.*

***Keywords:** Immune system, immune response, ordinary differential equations, numerical simulations.*

1. INTRODUCTION

The use of mathematical models for investigations of the behavior of immune system of organisms infected by pathogens such as viruses or organisms suffering from cancer, can be effective tool for determining the tendencies of the disease under medical treatments or without them.

An organism that meets a specific antigen for the first time possesses only a small amount of lymphocytes able to recognize and neutralize the pathogen.That is why the acquired immune system needs at least several days while bigger amount of specific lymphocytes are produced and activated. During this period of time the fight against the pathogen is performed by the innate immunity, which functions quickly but does not possess specificity and efficiency. As a result the infection can become strong and difficult to eradicate [1].

When foreign antigens enter an organism, both humoral and cellular types of acquired immunity start to function. Their mechanisms of functioning are different. The humoral immunity applies antibodies, which neutralize free viral particles. The cellular immunity system employs cytotoxic T lymphocytes (CTL), which destroy infected host cells [1].

In the paper we present a model, which is a generalization of a basic model proposed by G. Marchuk [3] and a model proposed by D. Wodarz [5]. In our model we assume that the growth of the virus depends on the amount of the free viral particles that have entered the organism. Additionally, we

suppose that the production of antibodies and CTL depends on the degree of the damage of the target organ: the higher is the damage, the weaker is the production of antibodies and CTL.

The purpose of this paper is to illustrate the application of mathematical and computational methods to immunology. The contents of our work are organized as follows. In Section 2 we describe our mathematical model of acquired immune response to viral infection. The model is a complicated system of ordinary differential equations. Theorems for existence, uniqueness and non-negativity of its solution are proved. In Section 3 we present some results of our simulations of primary and secondary immune response and comment their biological meaning.

2. MODEL DESCRIPTION

The interacting populations included in our model and their notations are the following:

x(t) - concentration of the susceptible uninfected cells of the target organ;
y(t) - concentration of the infected cells;
v(t) - concentration of the free virus particles;
z(t) - concentration of CTL specific for the virus;
w(t) - concentration of antibodies (immunoglobulins) specific for the virus;
m(t) - degree of the target organ damage.

The proposed model describing the time dynamics of the considered variables consists of the following six ordinary differential equations (ODE):

$$(1) \quad \frac{dx(t)}{dt} = l - d.x(t) - b_1.x(t).v(t)$$

$$(2) \quad \frac{dy(t)}{dt} = b_1.x(t).v(t) - a.y(t) - p.y(t).z(t)$$

$$(3) \quad \frac{dv(t)}{dt} = k.v(t) - q.v(t).w(t)$$

$$(4) \quad \frac{dw(t)}{dt} = N(m).g.v(t).w(t) - h.(w(t) - \overline{w}) - r.v(t).w(t)$$

$$(5) \quad \frac{dz(t)}{dt} = N(m).c.v(t).z(t) - b.(z(t) - \overline{z}) - f.y(t).z(t)$$

$$(6) \quad \frac{dm(t)}{dt} = s.v(t) - n.m(t).$$

We suppose that the parameters of the model (1)-(6) are non-negative constants and parameters l; $\overline{z}$ and $\overline{w}$ are positive. We look for solution such that the unknown functions are continuously differentiable with non-negative initial conditions.

Equation (1) describes the dynamics of the population of the susceptible uninfected cells. The meaning of its parameters is the following: l describes

the production of uninfected cells; d - the rate of decrease of uninfected cells due to their natural death; b_1 - the rate of decrease of uninfected cells due to their infection by virus.

Equation (2) describes the dynamics of the population of the infected cells. Parameter a characterizes the decrease of concentration of infected cells due to their natural death; p denotes the rate of decrease of infected cells due to their destruction by CTL.

Equation (3) describes the time dynamics of the concentration of free virus particles. The meaning of its parameters is the following: k denotes the rate of production of virus particles ; q - the decrease of virus particles due to their neutralization by antibodies.

Equation (4) describes the dynamics of the concentration of antibodies. Their production depends on the amount of viruses and on the degree of target organ damage. The meaning of its parameters is the following: g characterizes the production of antibodies; h - the rate of their natural death; r - the decrease of concentration of antibodies due to their antiviral activity; $\overline{w}$ is the amount of antibodies circulating in a healthy organism.

Equation (5) describes the dynamics of CTL. The meaning of its parameters is the following: c characterizes the production of CTL; b - the natural death of CTL; f - the decrease of concentration of CTL due to their killing activity against infected cells; $\overline{z}$ is the amount of CTL circulating in a healthy organism.

Equation (6) describes the degree of target organ damage. The damage depends on the amount of virus particles and can decrease due to recovery activity processes in the organism. The meaning of its parameters is the following: s denotes the rate of damage of the target organ by viruses; n - the rate of restoration of the target organ.

N(m) (participating in equations (4) and (5)) is assumed to be non-increasing non-negative continuous function that accounts for the violation of the normal functioning of the immune system due to the damage of the target organ [3]. We assume that there exists its limit value $\overline{m} \in (0,1)$. If the value of m is less than $\overline{m}$ we suppose that the damage of the infected organ is small and it does not affect the efficiency of the immune system. On the other hand, if m is greater than $\overline{m}$, we suppose that the damage of the infected organ is considerable and the immune response is weakened.

Adequacy of the model to real processes

Theorem 1: If the system (1)-(6) with initial conditions $x(t_0) = x_0 > 0$, $y(t_0) = y_0 \geq 0$, $v(t_0) = v_0 \geq 0$, $w(t_0) = w_0 = \overline{w} > 0$, $z(t_0) = z_0 = \overline{z} > 0$ $m(t_0) = m_0 \geq 0$ possesses solution then this solution is non-negative for every $t \geq 0$.

Proof: Let us assume that there exist values of $t > 0$ such that $x(t) < 0$. From the initial condition x(0) > 0 and the continuity of the function $x(t)$ it follows that there exists an instant in time t_1 at which $x(t)$ changes its sign i.e. $x(t_1) = 0$, and let t_1 is the smallest value of t, for which $x(t_1) = 0$. From here we would have $\frac{dx}{dt} \le 0$ when $t = t_1$. This would be a contradiction with equation (1) giving

$$\frac{dx}{dt}\Big|_{t=t_1} = l - d.x(t_1) - b_1.x(t_1).v(t_1) = l > 0$$

Therefore the assumption about the possible negativity of x(t) is incorrect.

From equation (3) follows $v(t) = v(0).e^{\int_0^t (k-q.w).dt} \ge 0$ when $t \ge 0$.

From the second equation is obtained

$$y(t) = e^{-\int_0^t (p.z(u)+a)du}\left[y(0) + \int_0^t b_1.x(u).v(u).e^{\int_0^t (p.z(u)+a)du} du \right] \ge 0 \text{ when } t \ge 0.$$

From equation (4) follows

$$w(t) = e^{\int_{00}^{tt} (N(m)g.v(p)-h-rv(p))dp}\left[w(0) + \int_0^t h.\overline{w}.e^{-\int_0^t (N(m).g.v(p)-h-r.v(p))dp} dp \right] \ge 0$$

when $t \ge 0$ because $w(0) \ge 0$.

From equation (5) follows

$$z = e^{\int_0^t N(m).c.v(u)-b-f.y(u))du}\left[z(0) + \int_0^t b.\overline{z}.e^{-\int_0^t (N(m).c.v(u)-b-f.y(u))du} du \right] \ge 0.$$

From equation (6) follows

$$m = e^{-n.t}\left[m(0) + \int_0^t s.v(p).e^{np} dp \right] \ge 0 \text{ when } t \ge 0.$$

Theorem 2. For every T > 0 on the interval [0; T] there exists an unique continuously differentiable solution to the system (1) - (6) with initial conditions $x(t_0) = x_0 > 0$, $y(t_0) = y_0 \ge 0$, $v(t_0) = v_0 \ge 0$, $w(t_0) = w_0 = \overline{w} > 0$, $z(t_0) = z_0 = \overline{z} > 0$, $m(t_0) = m_0 \ge 0$.

Proof. The local existence of the solution follows from the continuity of the righthand sides [2]. The uniqueness of the solution follows from the

continuity of the partial derivatives of the right-hand sides with respect to the unknown functions [2].

The functions x(t), y(t), v(t), w(t), z(t) and m(t) are bounded on [0; T], therefore they receive their maximal and minimal values on [0; T]. It is trivial to prove that the nonlinear system (1) - (6) behaves not worse than a linear system. Therefore a global solution on [0; T] exists.

3. INVESTIGATION OF THE PRIMARY AND SECONDARY IMMUNE RESPONSE TO VIRAL INFECTIONS. NUMERICAL SIMULATIONS

Organism that has never met an antigen inevitably has only one or a few lymphocytes that can recognize this antigen. After the destruction of the antigen by the primary immune response (fig.1), most lymphocytes recognizing the antigen, died. The best of these lymphocytes remain as immune memory cells and at the next meeting with the same antigen already secondary immune response is much faster and more efficient. It is often so effective that the disease does not appear. This explains the illness of some infections only once. The power and efficiency of the secondary immune response due to the strong start (fig. 2). These advantages are applied in the preparation of vaccines.

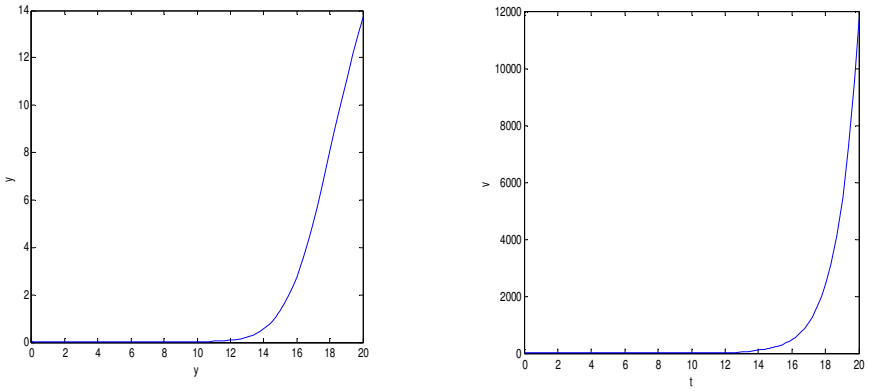

Fig 1. *Dynamics of the concentration of the infected cells and viruses in the primary immune response.*

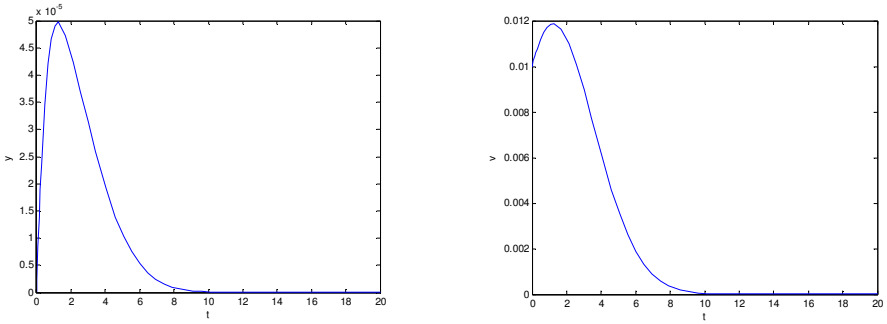

Fig. 2. *Dynamics of the concentration of the infected cells and viruses in the secondary immune response*

The numerical simulations were made with ode15s of Matlab and RelTol=1.e^{-3} и AbsTol=1.e^{-6} that provides precision 0.001 for each point in time [4]. The parameter and initial values are chosen as follows : l=10, d=10, b1=0.01, a=0.5, k=0.8, u=1, q=0.5, g=10, h=0.17, r=10, s=10, n=0.12, c=10, w1=5, b=0.2, f=5, z1=5, p=1, x(0)=1, y(0)=0, v(0)=0.01, w(0)=1, z(0)=1, m(0)=0.

4. CONCLUSIONS

In the paper we have used a mathematical model to study some features of the competition between virus and adaptive immune system. The model has been solved numerically. Our future research plans are related to application of the model to clinical data and development of the model for more detailed analysis of the adaptive immune reaction to viral infections.

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Investigation of the Role of Matrix Degradative Enzymes on the Migration of Tumor Cells

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***Abstract**: The paper is devoted to numerical investigation of the influence of the ability of matrix degradative enzymes (MDE) to destroy the extracellular matrix (ECM) on the migration of tumor cells through healthy tissue. The corresponding mathematical model is a system of three differential equations describing the dynamics of tumor cells, ECM and MDE.*

***Keywords:** Mathematical model, numerical simulation, tumor, partial differential equations.*

1. INTRODUCTION

The mathematical modeling is among the main tools used in scientific investigations of biological processes. Over the recent decades it has been actively applied in immunology, cancer research and other areas related to human health. Often the qualitative and quantitative analyses of mathematical models allow to determine the main factors in the interactions between pathogens and immune system of human organism, their significance and to predict the probable course of the disease. Studies conducted by the use of mathematical models can sometimes decrease the need of medical, clinical or laboratory investigations, which usually are expensive and sometimes problematic from ethical point of view.

In our paper we present results of our numerical experiments with a model describing the interactions between tumor cells, MDE and ECM. We use the nonstandard finite difference scheme proposed in [3].

2. MATHEMATICAL MODEL

We consider the mathematical model proposed by Chaplain *et al.* [1, 2]. It is formulated as the following system of reaction-diffusion-chemotaxis equations:

$$
\begin{aligned}
&\frac{\partial n}{\partial t} = d_n \frac{\partial^2 n}{\partial x^2} - \chi \frac{\partial}{\partial x}\left(n \frac{\partial m}{\partial x} \right) + \mu_1 n(1-n-f), \\
(1)\quad &\frac{\partial f}{\partial t} = -\eta m f + \mu_2 (1-n-f), \\
&\frac{\partial m}{\partial t} = d_m \frac{\partial^2 m}{\partial x^2} + \alpha n(1-n) - \beta m.
\end{aligned}
$$

The model describes the interactions between tumor cells, whose density is denoted by $n = n(x;t)$, ECM, whose density is denoted by $f = f(x;t)$ and MDE, whose concentration is denoted by $m = m(x;t)$. The unknown functions are supposed to depend on the space variable x, which belongs to the scaled domain $\Omega = [0,1]$ of tissue, and time t. The meaning of the parameters of the model is as follows. The coefficients of diffusion d_n and d_m describe the random motility of tumor cells and MDE's respectively, the chemotactic coefficient χ - the directional motility of tumor cells toward higher concentrations of soluble MDE, μ_1 denotes the proliferation rate of tumor cells, η - the rate of destruction of ECM due to degrading functions of MDE, μ_2 - the reestablishment rate of ECM, α - the production of MDE by cancer cells and β - the rate of decay of MDE's. All parameters of the model are supposed to be non-negative.

The system (1) is supplemented by the zero-flux boundary conditions

$$
(2)\quad \frac{\partial n}{\partial x}(x,t) = \frac{\gamma}{d_n} n(x,t) \frac{\partial m}{\partial x}(x,t), \frac{\partial m}{\partial x}(x,t) = 0, x = 0,1,
$$

and initial conditions

$$
(3)\quad n(x,0) = n_0(x), f(x,0) = f_0(x), m(x,0) = m_0(x).
$$

The functions $n_0(x)$, $f_0(x)$ and $m_0(x)$ are supposed to be non-negative and not identically zero.

The model described by Eqs.(1)-(3) is analyzed numerically in [3] where a nonstandard finite difference method for its approximation is proposed. This nonstandard approximation is based on the rules proposed by Mickens [4]. One of these rules is to apply nonlocal representation for the approximation of the nonlinear non-derivative terms, i.e. to use approximate values of functions from different time or space levels on the mesh. Another rule is to use nonstandard approximations for the time derivatives [3], [4]. Using these suggestions, the authors of [3] have obtained an efficient numerical scheme

for solving the model problem described by Eqs.(1)-(3). Moreover, the obtained in [KG] approximation scheme guarantees the preservation of the non-negativity of the solutions, which is not always possible with conventional finite difference schemes.

3. NUMERICAL EXPERIMENTS

In this Section we present some of the results of our numerical experiments. We have studied the role of parameter η describing the rate of destruction of ECM due to the degrading activity of MDE. The system of algebraic equations obtained after the discretization of the model (1)-(3) has been solved by the use of Matlab. Following [1, 2], we choose the initial conditions, which are shown in Figure (1) and are defined as follows:

$$n(x,0) = \exp\left(\frac{-x^2}{\varepsilon}\right), f(x,0) = 1 - n(x,0), m(x,0) = 0.5n(x,0), \forall x \in [0,1].$$

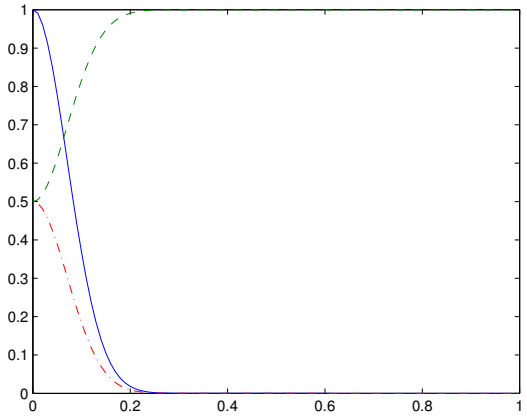

Figure 1. The initial distribution of tumor cell density (solid), ECM density (dashed), and MDE concentration (dashdot).

The obtained approximate solution for tumor cells, ECM and MDE for values of time $t = 2$ and $t = 5$ are presented in Figures (2)-(3) for value $\eta = 1$ and in Figures (4)-(5) for $\eta = 100$. The values of the remaining parameters have been set to: $\alpha = 0.4$, $\beta = 0.5$, $\chi = 0.02$, $\eta = 10$, $d_n = 0.01$, $d_m = 0.01$, $\mu_1 = 1.0$, $\mu_2 = 2.5$, $\varepsilon = 0.005$, $N = 80$.

The chosen different values of parameter η allow us to study the role of the destruction rate of ECM by MDE. In the first part of our numerical experiments we consider the case with low rate of ECM destruction setting

$\eta = 1$, while in the second part we consider the case with higher rate of ECM destruction setting $\eta = 100$.

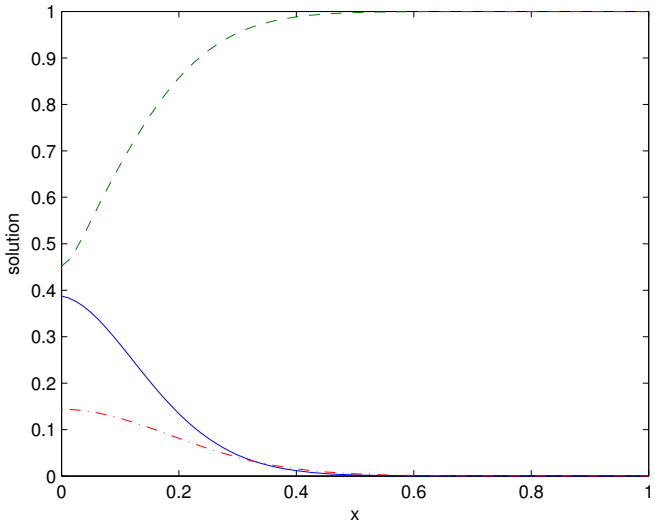

Figure 2. Values of tumor cell density (solid), ECM density (dashed), and MDE concentration (dashdot). Solutions with parameter value $\eta = 1$ at time $t = 2$.

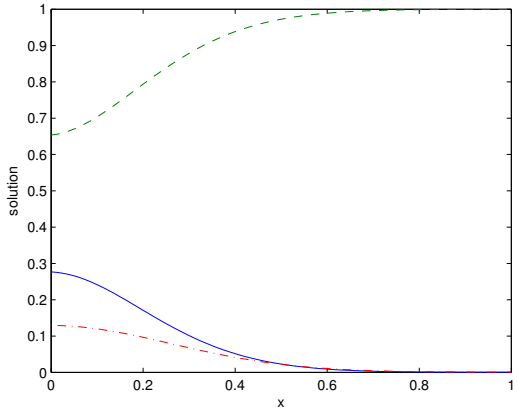

Figure 3. Solutions with parameter value $\eta = 1$ at time $t = 5$.

The comparison between the cases with low and high rate of ECM destruction shows that the higher degree of ECM destruction resulting from the high activity of MDE produced by cancer cells leads to deeper migration of cancer into the healthy tissue as well as to higher cluster of cancer cells.

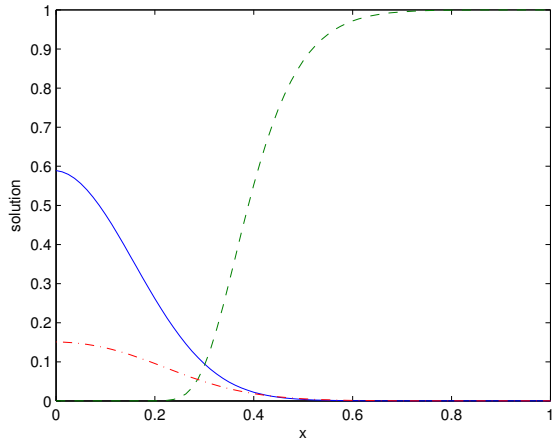

Figure 4. Solutions with parameter value $\eta = 100$ at time $t = 2$.

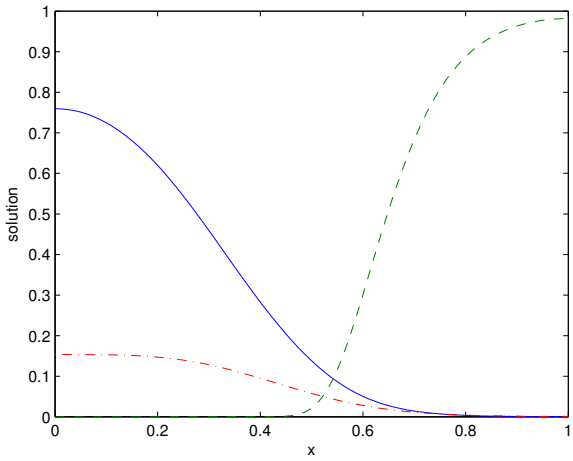

Figure 5. Solutions with parameter value $\eta = 100$ at time $t = 5$.

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A Comparison of Block Bootstrap Procedures for Periodic Time Series

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Abstract: *The bootstrap was resulted an useful method in estimating the distribution of an estimator or a test statistic by resampling the data in the case of independent and identically distributed observations. Although it was not as efective in the case of dependent data as in the case of independent and identically distributed data, an adaptation was obtained using the block bootstrap. The block bootstrap consist in dividing the data into blocks of observations and then resampling these blocks with replacement. Several block bootstrap methods were proposed, using overlapping blocks or nonoverlapping blocks. When resampling periodic data, we must take in consideration the periodicity present. The traditional block bootstrap procedures do not give the pretended results. With intention to take in consideration the periodicity present in the data, different methods were proposed. In this paper we will use different block bootstrap procedures in our simulation of periodic data, with intention to make a simulation comparison of the performance of these block bootstrap procedures.*

Keywords: *Block bootstrap, periodically correlated time series, resampling, seasonality.*

1. INTRODUCTION

Since the introduction by Efron, bootstrap was resulted to be an important method for estimating the distribution of an estimator or test statistic applying the resampling of the data.

This method gives good results in the case when the observations are independent and indentically distributed (i.i.d), but in the case of dependent observations, like time series, the i.i.d bootstrap gives incorrect answers (Singh (1981), Babu and Singh (1983)). The block bootstrap resulted to be an important and well-known method for implementing the bootstrap with the time series data. This method consists in dividing the data into blocks of observations and then resampling these blocks with replacement. The block length is another issue that must be taken in consideration when we use a block bootstrap procedure. Although Kunsch (1989) provided theoretical

justification that the optimal block length to minimize the mean square error of the estimated variance is in the order of $N^{1/3}$, where *N* is the total number of observations, the theoretical criterion is not practically sufficient because generally in practice, the block length and its sensitivity must be studied empirically.

2. PERIODIC TIME SERIES

A time series X_t is a seasonal (periodic) time series with period T if X_t and X_{t+T} have the same distribution. Seasonal time series are time series in which similiarities occur at equivalent parts of a cycle.

A special class of seasonal time series are Periodically Correlated time series, also known as Cyclostationary time series. The time series X_t is called Periodically Correlated with period *T* if the mean and the covariance of the series remains the same when shifted *T* units of time. So, X_t is periodically correlated if:

$E(X_{t+T}) = E(X_t)$, $\operatorname{cov}(X_{t+T}, X_{s+T}) = \operatorname{cov}(X_t, X_s)$ for all integers *s* and *t*.

The class of Periodically Correlated time series is quite large. Many processes encountered in nature arise from periodic phenomena and the notion related to these processes seems to initiate with Bennett (1958) who called them cyclostationary processes.

Periodically correlated models with nonzero seasonal mean are of the form $X_t = Y_t + \mu_t$ where Y_t is a periodic time series with period *T* and zero mean and $\mu_t, t \in Z$ is a deterministic periodic function with period *T*.

3. BOOTSTRAP PROCEDURES FOR PERIODIC TIME SERIES

Several block bootstrap methods were presented from researchers with intention to estimate the parameters of interest. Therefore, when we study periodically correlated data, these methods must be able to considerate the periodicity present. There is a large amount of block bootstrap methods for the time series data, but not as much for the time series with a seasonal component.

In the case of a time series with a seasonal component, Politis (2002) proposed a resampling algorithm that takes in consideration the periodicity present. The Seasonal Block Bootstrap (SBB) proposed by Politis is a version of Kunsch's (1989) block bootstrap with blocks whose size and starting points are restricted to be integer multiples of the period. Although the good results of SBB presented by Politis, the method posess a restriction on the relative

size of the period and the block size, because the block size must be of the order of the period.

Chan et. al. (2004) proposed a version of block bootstrap for the time series with periodicity that works by making a partition of the period into small time intervals and divide the periodic data into blocks that correspond temporally to each of the intervals and then independently resample with replacement from the blocks from each interval and form a new time series of bootstrap data. However this resampling procedure was designed for the periodic time series with long periodicities, because it is assumed that the block length is smaller compared to the period. For the consistency of this block bootstrap procedure is required that the period length increase infinitely as the sample size increases. This is the reason that in the time series with fixed-length periodicity the block bootstrap procedure proposed by Chan et. al. (2004) is not consistent.

Dudek et. al (2013) propose a modification of the block bootstrap that is suitable for periodic time series with fixed length periodicities of arbitrary size as related to block size and also to the sample size. The series of observations is divided in blocks of desired length independent from periodicity and then these blocks are resampled in a way that retains periodicity. The block size can be chosen independenly from the period,so the usual asymptotic considerations for block size choice avoid the problems that associate the method proposed by Chan et al. (2004) and also the Seasonal Block Bootstrap of Politis. The GSBB preserve the periodic structure of the data not requiring any particular relationship between the block length and the period length. This method is consistent for the seasonal means and the overall mean of a periodically correlated time series.

Dudek et al. (2014) also demonstrated that the actual coverage probabilities for the bootstrap simultaneous confidence intervals constructed, are very close to nominal ones for the wide range of the block length choice.

A block bootstrap method for periodic time series, that is resulted as a real competitor of other block bootstrap procedures, is the block bootstrap resampling of the ‘residuals’ (BBR). It consists in using block bootstrap resampling in the residuals of the original series after the seasonal means are estimated and removed.

Even in this proposed resampling method, choosing a block length optimally is always a difficult problem. Different methods for block size choice in the context of stationary data are described in Nordman and Lahiri (2012), but some of this methods are applicable after the seasonal components have been estimated and removed. The degree of overlap among blocks to be bootstrapped plays a major role in efficiency because maximum overlap leads to maximum efficiency.

The BBR method resulted an efficient method in some models of Periodically Correlated time series compared with GSBB.

4. SIMULATIONS

Our goal is to realize a comparison of the results using algorithms of the two above mentioned methods, GSBB and BBR. We choose our model of a seasonal time series:

$$X_t = \sin(2\pi t / T) + \cos(2\pi t / T)e_t \text{ with } e_t \sim N(0,1)$$

It is obvious that this model is a model of a periodically correlated time series with period T. We conducted a number of related simulations with 1000 Monte Carlo trials and B=500 bootstrap replications. The period chosen is T=12, we take two values N=120 and N=240 and also the chosen block lengths are from the set {2, 3, 4, …, 30}. The simulations are used to construct the bootstrap equal-tailed pointwise confidence intervals for the overall mean of the above time series. We calculated the Actual Coverage Probability (ACP) using the Generalized Seasonal Block Bootstrap (GSBB) and Block Bootstrap of Residuals (BBR) with the Nominal Coverage Probability (NCP) 0.95.

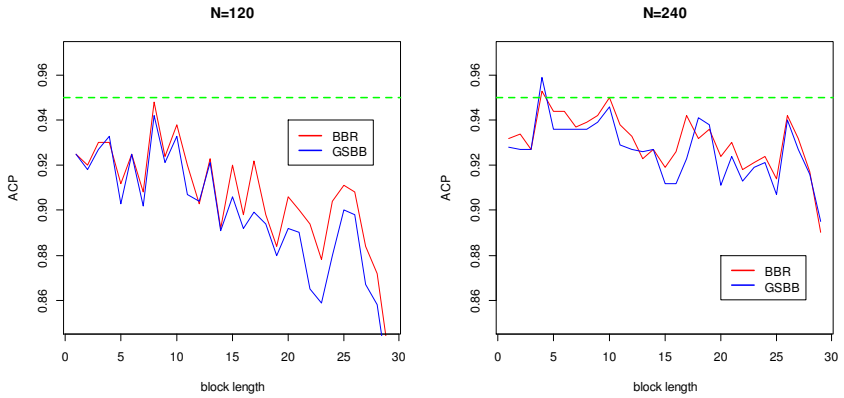

Fig.1 ACP for the proposed model, N=120 (left) and N=240 (right), $b \in \{2,3,...30\}$ and the NCP (the green line).

It seems that for both N=120 and N=240 the ACPs are mostly higher using BBR than GSBB in our proposed model. The results for the ACP in the case of N=240 are better than N=120 for both methods. The optimal block length choice is b=9 for BBR and GSBB for N=120, that gives ACP=0.948 (94.8%) and ACP=0.942 (94.2%). For N=240, the optimal block length is b=11 and ACP=0.95 (95%) for BBR and ACP=0.946 (94.6%) for GSBB.

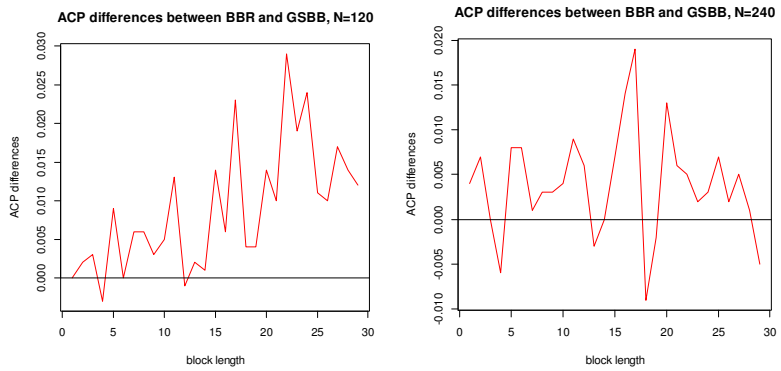

Fig.2 Differences between ACPs for BBR and GSBB, N=120 (left) and N=240 (right), for block length $b \in \{2,3,...30\}$

The difference between ACP results from BBR and GSBB for N=120 and N=240 it is mostly positive, demonstrating an advantage if we use BBR algorithm.

5. CONCLUSIONS

In this paper we provide a theoretical view of several block bootstrap methods for time series with a seasonal component and also a simulation comparison between two block bootstrap algorithms. GSBB resulted a suitable method in many cases of time series with periodic structure. BBR enjoys good properties as well.

Although in our proposed model of a Periodically Correlated time series, the results obtained using BBR outperform those obtained from the GSBB.

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Using CLIPS to Realize Quality of Life Expert System (QLIFEX)

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***Abstract**: This article presents an architectural framework of an expert system for social area, as describing the architecture, design, development and implementation of the expert system. The development of an expert system for quality of life evaluation is a novel information technology derived from research in artificial intelligence. The unique contribution is that, unlike the other methods for assessing the quality of life using the quantitative data, this expert system uses qualitative data only. The system represented in this article is called QLIFEX (Quality of Life Expert System), and it is designed for providing an expert assessment of quality of life in the social field. In its implementation, C Language Integrated Production System CLIPS has been used.*

***Keywords:** expert systems, knowledge acquisition and management, quality of life, CLIPS.*

1. INTRODUCTION

One of the most important challenges the European Union was facing at the beginning of the 21st century was to balance economic development with the improvement of quality of its citizens life. Assessing the quality of life is a pivotal problem when it comes to the social field in which the design of such a problem usually requires a lot of time. Each model or system that helps a great deal to solve that problem could be useful for establishing a successful assessment of quality of life.

Expert systems are a part of artificial intelligence. An expert system allows the transfer of specialized knowledge from experts to computer programs. The expert systems can be categorized into three main groups: rule-based; inductive; hybrid.

It is good for society to use expert systems when solving one of the main problems of the domain - social area: what is the degree of quality of life? In

the article, the development process of the expert system QLIFEX (Quality of Life Expert System), which uses various parameters – indicators, factors, and an uncertainty factor, is revealed in detail, and is shown that those parameters are defined by the constructor of the system in order to solve the problem raised, namely to determine the quality of life of people. QLIFEX is a rule-based expert system, which collects a set of factual data (indicators) and produces an activity (determines the degree of quality of life) by interpreting the data with the help of a set of pre-defined rules (rules for determining values of the factors).

Previous studies, related to research and evaluation of quality of life, contain one or more of the following disadvantages: The research studies only using quantitative methods; The studies on quantitative data only; Sometimes emotional aspects could affect the researchers' work; The data are not subjective but objective (the subjective evaluation of people should be taken into account as well).

To overcome the above drawbacks, the expert system QLIFEX was set up. This expert system is useful for those users who are not specialists in the social sphere and cannot rely on specialists from it in their research work. It is particularly suitable for and can be used in the education and training of students from different faculties and disciplines. When students are majoring in Computer Science and Mathematics the focus should be on the development and evaluation of the system, while the attention of students in humanities can be directed to the applied aspects of the expert system itself [1].

Since C-like Language for Integrated Production System (CLIPS) is proven to be suitable for expert system development that is why this language has been chosen to build an expert system for evaluation of the quality of life.

2. DESING AND METHODOLIGY

After several prototypes, a decision was made that in the design and development process of this expert system, CLIPS: Expert System Shell was going to be used [2]. Shell has got facilities in terms of the questions in the knowledge base. It contains rules presented in its own syntax for its knowledge base. It consists of facilities that are used to write the rules, which build the knowledge base.

2.1. CLIPS

The language that is used for knowledge representation is C Language Integrated Production System (CLIPS), which is a programming language [4] developed in NASA's Johnson Space Center in the 80s of the 20th century. It is an easily accessible public software that is well-documented [5]. CLIPS

is very flexible, allowing the calling of external functions to be transported across different operation systems. It inherits all the features of previous languages like List Processor (LISP) and PROgramming in LOGic (PROLOG) and adds a new, unique ability to combine rules with objects. CLIPS environment is Windows-based, which provides good opportunities for bug tracking. In addition, all activities can be seen in separate windows during the execution of the program. These windows provide transparency of the system so that the process of decision-making can be traced. In CLIPS rules with the highest "salience" value are executed first. Thus the language provides actions to facilitate the control of execution of the sequence of the rules.

CLIPS has no built-in capabilities for managing uncertainty. However, it is possible to incorporate uncertainty in CLIPS by incorporating information dealing with the uncertainty directly into facts and rules (uncertainty factor).

The main elements of CLIPS are: Fact-list: the global memory for data, which contain the facts; Knowledge-base: contains all the rules; Inference engine: forward inference.

2.2. Architecture

The main components of an expert system are the knowledge base and the interface module, which provides communication amongst the user, the system and the interpreter.

The knowledge base of the expert system QLIFEX contains facts describing the domain (the degree of quality of life is depending on the values of the factors) and the logical interrelationship amongst those facts. In the core of the knowledge base are positioned the rules, which define what should be done in a particular situation (based on those rules the values of the factors are calculated by the values of the indicators).

The interpreter performs in a specific order while processing the knowledge derived from the knowledge base.

Knowledge processing = Sequential consideration of all the rules
that belong to an expert system

If the condition is satisfied → Execution of a specific action → Proposal for a possible solution

This sequence of actions in QLIFEX is: If a condition is true → The values of the factors are defined by the values of the indicators as using the mechanism "uncertainty factor" → Based on the values of the factors, as a result, is determined the degree of quality of life.

2.3. CLIPS facts and rules

An expert system relies on a set of facts and a set of rules to provide a solution. In QLIFEX the facts and the rules are formed in a static manner,

according to the knowledge of experts from the various subareas of social sphere. The facts of QLIFEX include various degrees from the ten-degree global scale for quality of life, as these degrees are defined by the values of the factors [3]. The CLIPS facts of QLIFEX look as follows:

```
(deffacts QLIFES::the-qlifequal-lawist
 (qlife (name Perfect-quality-of-life)
 (econ-situation very-stable)
 (hous-environment completely-satisfied)
 (employ-education very-good-condition)
 (fami-relation harmonious-relationships)
 (work-life-balance well-balanced)
 (health-care completely-accessible)
 (urban-subject absolute-optimism)
 (concepts-quality-society very-satisfied))
```

Knowledge is almost always incomplete and / or uncertain. The set of methods for using uncertain knowledge in combination with the uncertain data in the process of reasoning is called uncertainty reasoning. The uncertain knowledge it can handle with as putting them into rules with uncertainty factor or with weight. When building the knowledge base of the system QLIFEX, a special attention was being paid to the presentation of the uncertain knowledge. In the case of the expert system QLIFEX, it was used an additional technique called uncertainty factor; its rate is: form 0% to 100%.

The QLIFEX rules are a list of actions waiting to be triggered when the required conditions are satisfied. In general, a "IF - THEN" production rule from the knowledge base can be represented as follows:

(if precondition$_1$ and precondition$_2$ and ... and precondition$_n$)
(then conclusion$_1$ and/or conclusion$_2$ and/or ... and/or conclusion$_m$)

The rules consist of two parts: an assumption (IF) and a conclusion (THEN), each of which contains combinations of lower detailed level expressions. The following program fragment is an example of a production rule of the expert system QLIFEX:

```
(rule (if right-privacy is no and  additional-training is
          maybe and  children-home is sometimes)
      (then best-work-life-balance is balanced with certainty
          40))
```

2.4. User Interface

The main function of the interactive user interface is to provide communication with the environment and to convert information from the outside world for the internal representation, and vice versa. It is used to enter the values of the indicators and to display the final result, namely the degree

of quality of life. Since all the data used in QLIFEX are quality, the method that is used to enter information is a natural language. In the implementation process of QLIFEX, it has been used the opportunity to receive information not only in the form of solutions, but also in the form of explanations.

3. IMPLEMENTATION

The implementation of QLIFEX consists of four basic steps: setting up the module with questions - QLIFE-QUESTIONS; defining the module with facts - QLIFES; defining the module with rules - QLIFE-RULES; defining the modules for managing and implementing the control transfer in accordance with the user answer - QUESTIONS RULES and THE RULES.

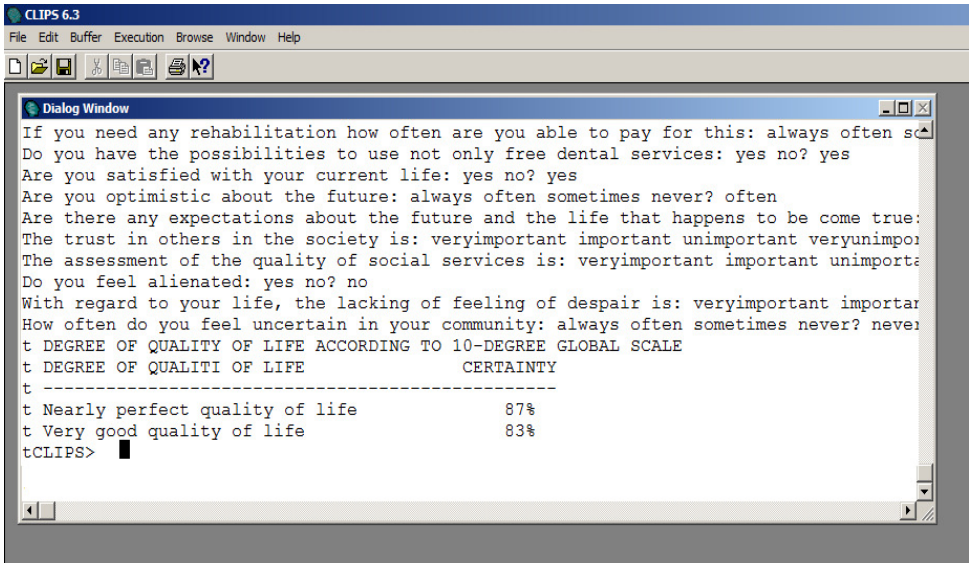

Fig. 1: The CLIPS dialog window for interactive input of the values of the indicators, and the display of the output - degree of quality of life

CLIPS provides a command line interface (CLI), while application programming Interface (API) can be achieved when an expert system embeds in C++ application or Java application. Once the inference engine of CLIPS has produced a result, the expert system will display it, and this result equals one degree from the ten-degree global scale for quality of life (Figure 1).

4. CONCLUSIONS

The expert system QLIFEX is an innovative research project based entirely on qualitative methods, which aims at helping in understanding of how in an era of great changes residents from different countries live and work in diverse economic organizations, and how they would rate their work and life.

The new expert system QLIFEX is designed for a specific purpose – quality of life evaluation. The importance of this expert system lies in the fact that it is not specific to one area and covers a set of subareas of the social area domain - Economic situation, Housing and environment, Employment, education and skills, Structure of the household and family relationships, Balance between work and life, Health and healthcare, Urban subjective, Concepts of the quality of society.

5. ACKNOWLEDGMENTS

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Surface Fitting of the Results of Molecular Docking and the Biological Activity of Delta Opioid Selective Enkephalin Analogues

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***Abstract**: Activation of delta opioid receptor system produces a behavior profile distinct from that of other opioids, therefore the design of new delta selective agonists is actual. The quantitative structure-activity relationship can be predicted using different mathematical models. The aim of the presented study is to find a relationship between structures of both-receptor and ligand, characterizing biological activity.The relationship of the quantitative parameter of in vitro bioassay with the data from the docking was modeled with first to third degree polynomials using the least squares method. It was established that the polynomial of the third degree has the best fit, so it could be applied for determination of the quantitative structure-biological activity relationship.*

***Keywords:** Surface fitting, Computer modeling, Docking, Delta opioid receptor, Affinity, Efficacy, Potency.*

1. INTRODUCTION

Enkephalins are endogenous opioid pentapeptides [1,2,3] with delta-opioid receptor (DOR) preferences [4], but they cannot be used directly for therapy as they are rapidly enzymatically degradable and cannot pass through biological membranes. The discovery of new potent and selective ligands to the DOR is important. New *in silico* methods allow to shorten this process by simulating ligand-receptor interactions.

The aim of the current study is to apply the model that describes relationship between the biological activity of delta-selective enkephalin analogues and docking results. In order to achieve this purpose the following problems should be solved: (1) performance of molecular docking calculations of homology modeling of DOR [5,6] and delta-selective enkephalin analogues, and also calculation of the total energies of ligand-

receptor complex after docking experiments; and (2) finding a function $z = f(x, y)$ from some class polynomials, fitting in hest way given m distinct data points $\{(x_i, y_i, z_i)\}_{i=1}^{m}$ in R^3.

2. METHODS

2.1. Objects

A model of DOR obtained by method of homology modeling was used. This model is named Model B and it was presented in our previous study [5,6]. Eleven ligands, investigated for their potency, selectivity and efficacy to DOR with *in vitro* bioassay in a previous study [7-9] were selected for docking experiments.

2.2. Docking procedure

The docking experiments were carried out with software GOLD 5.2, which uses a genetic algorithm **[10-12].** In this article it was described the implementation of the *Astex Statistical Potential (ASP)* [13] function as a scoring function for GOLD 5.2 and its usefulness to perform docking precisely, to predict the binding energies, and to realise the biological effects of investigated compounds. It differs from other statistical potentials by the choice of the so-called reference state*,* which determines how the raw distribution of observations is transformed into potentials. The contacts between atoms are usually determined by radial distribution functions (RDFs). Given an atom at some position the RDF will tell us how many other atoms we can expect to find at a distance between *r* to *r+dr*, where *dr* is the bin width in the RDF and can be thought of as the 'thickness' of a spherical shell. A statistical potential (Eq.1) between two atom types *i* and *j* is defined as:

$$(1) \quad StatScore_{(i,j,r)} = -ln\frac{n_{obs^{(i,j,r)}}}{n_{exp}^{StatScore_{(i,j,r)}}}$$

$$n_{exp}^{ASP} = \langle \frac{n_{obs^{(i,j,r)}}}{f_p(i,r')f_l(j,r'4pr'^2Dr)} \rangle_{r'=6.0}^{r'=8.0} \partial f_p(i,r')\partial f_l(j,r')\partial 4pr^2Dr$$

$$ASP\ Fitness = -C_s f_p f_l StateScore(p,l,r_{pl}) - c_{int}E_{int} - c_{clash}E_{clash}$$

where f_p – volume corrections to the contacts of protein; f_l – volume corrections to the contacts of protein ligand; r – volume of a spherical shell at distance; r_{pl} – distance between protein atom p and ligand atom l; C_s - scaling factor; C_{int} - internal energy; C_{clash} - clash coefficients.

The total energies for obtained ligand-receptor complex were calculated by Molegro Molecular Viewer (MMV Version 2.5) [14]. For analyzing the docking results it was used Ligand Energy Inspector tool of MMV. In order to calculate the total energy of the ligand-receptor complexes, it was used MolDock scoring function (Eq. 2).

$$(2) \qquad E_{score} = E_{inter} + E_{intra}$$

where E_{score} is a docking scoring function, E_{inter}-ligand-protein interaction energy, and E_{intra} - internal energy of the ligand.

2.3. MATLAB

The fitting of experimental data is performed by the polynomial function (Eq.3):

$$(3) \qquad z = \sum_{0 \le i+j \le n} a_{ij} x^i y^j,$$

where z is the potency of the ligand; x is the scoring function; y is the total energy of ligand-receptor complex; *n* is the degree of the polynomial; a_{ij} are the parameters of the model. The coefficients of Eq.3 were determined by the least square method, it means

$$(4) \underset{(a_{00},\dots,a_{0n})}{\text{minimize}} F(a_{00}, \dots, a_{0n}) = \sum_{s=1}^{m} \left(z_s - \sum_{0 \le i+j \le n} a_{ij} x_s^i y_s^j \right)^2,$$

m is the number of ligand-receptor complexes (data points).The values of z, $z_1, z_2 \dots, z_n$ represent the values of potency (*in vitro* parameter); the values of x, $x_1, x_2 \dots, x_n$ represent the result from the docking (scoring function); the values of y, $y_1, y_2 \dots, y_n$ represent the total energy for ligand-receptor complex.To study the fitting behaviour of several polynomial degree functions, it has been performed series of fittings, starting from first to third degree. The Surface Fitting Tool of MATLAB [15] was applied and the individual model could be interpreted as a surface fitting function of the experimental data by least squares method (Eq.4).

- *Parameters to evaluate the goodness of fit*

- *SSE (Sum of squares due to error)* - the total deviation of the response values from the curve fit to the response values (Eq.5). A SSE value near to 0 shows that the model has a smaller random error component and then the fit will be more useful for prediction.

$$(5) \quad SSE = \sum_{i=1}^{n} \omega_i (y_i - \hat{y}_i)^2$$

Here y_i is the observed data value, $\hat{y}_i$ is the predicted value from the curve fitting and ω_i is the weighting applied to each data point, here ω_i=1.

- r^2 - measures how successful the fit is in explaining the variation of the data (Eq.6); it is defined as the ratio of the sum of squares of the regression (SSR) and the total sum of squares (SST) about the mean (Eq.7). r^2 can take on any value between 0 and 1, with a value closer to 1 indicating that a greater proportion of variance is accounted for by the model.

$$(6) \quad r^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

$$(7) \quad SSR = \sum_{i=1}^{n} \omega_i (\hat{y}_i - \bar{y}_i)^2 , SST = \sum_{i=1}^{n} \omega_i (y_i - \bar{y}_i)^2, \; SST = SSR + SSE$$

- *Adjusted* r^2 (Eq. 8)-the best indicator of the fit quality when compare two models. It can take on any value less than or equal to 1, with a value closer to 1 indicating a better fit.

$$(8) \quad adjusted\, r^2 = 1 - \frac{SSE(n-1)}{SST(v)}$$

- *RMSE (Root Mean Squared Error)* - the standard error of the regression (Eq.9). A RMSE value closer to 0 indicates a fit that is more useful for prediction.

$$(9) \quad RMSE = s = \sqrt{MSE}$$

MSE - the mean square error $(MSE = SSE/v)$.

- *Degrees of Freedom* - the number of response values minus the number of the fitted coefficients estimated from the response values.

3. RESULTS AND DISCUSSION

3.1. Docking results and calculation of total energy

For surface fitting of the relationship between biological activity of enkephalin analogues, total energy calculates by MMV and ASP scoring function were applied with methods described in Section 2. The results are presented in Tab.1. Docking was performed with the DOR obtained by homology modeling and all 11 ligands in GOLD 5.2 [**11,12**]. The binding site of the receptor is known from the literature [16]. It is the residues within 10 Å around an aspartic acid residue (Asp128). The ASP scoring function is used to rank these ligand orientations/conformations by evaluating the binding density of each of the probable complexes. The ligand-receptor complex between Model B of DOR and ligand DPDPE is shown in Fig.1 as an example. The results of the docking (Tab.1) show the relative pose prediction performance of GOLD 5.2 by the ASP scoring function [17].

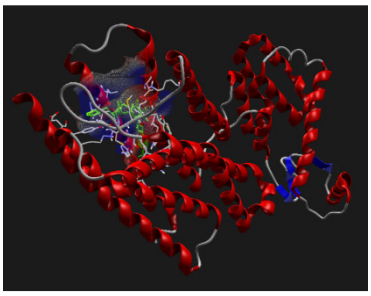

Fig.1: Diagram of ligand-receptor complex between Model B of DOR and ligand DPDPE. The receptor is presented in ribbons and helixes. This diagram was generated with the MMV.

Tab 1: The values of the main parameters used for surface fitting: ASP scoring function calculated by GOLD 5.2, total energy calculated by MMV and IC_{50} obtained by *in vitro* bioassay.

Ligands	ASP function	Total energy	IC_{50}
[Cys(Bzl)^2-Leu^5]-enk	20,26	-77,135	8.3
[Cys(Bzl)^2-Met^5]-enk	25,16	-98,91	9.53
[Cys(O_2NH_2)^2-Leu^5]-enk	22,66	-99,678	1.29
[Cys(O_2NH_2)^2-Met^5]-enk	26,18	-88,498	2.22
[DCys(O_2NH_2)^2-Leu^5]-enk	24,31	-66,115	11.4
[DCys(O_2NH_2)^2-Met^5]-enk	-12,82	897,265	75.96
DPDPE	19,58	-75,943	6.18
[HCys(O_2NH_2)^2-Leu^5]-enk	18,87	-90,567	31.92
[HCys(O_2NH_2)^2-Met^5]-enk	23,84	-80,137	16.09
[Leu^5]-enkephalin	22,45	-104,149	11.45
[Met^5]-enkephalin	33,9	-112,752	18.91

As it can be seen from the results in Tab. 2 the goodness of fit statistics shows that the obtained polynomial model for fitting of the data with the third degree is a good one. It is with the highest value of $r^2 = 1.0$ and the value closer to 1 indicating that a greater proportion of variance is explained by the model. Therefore this model explains a high proportion of the variability in experimental data, and is able to predict new observations with high variability.

Tab. 2: Assessing the goodness of fit for the polynomial models obtained by least squares method.

Degree	SSE	r^2	Adj r^2	RMSE	№ Coefficients
First	752.844	0.8318	0.7897	9.7011	3
Second	287.3484	0.9358	0.8716	7.5809	6
Third	0.0246	1.0000	0.9999	0.1568	10

The best results for fitting data according to the results in Tab. 2 was obtained for surface fitting by a cubic polynomial in 3D for determining the relationship between potency and docking results (Eq.(10)):

$$(10)\ f(x,y) = a_{00} + a_{10} * x + a_{01} * y + a_{20} * x^2 + a_{11} * x * y + a_{02} * y^2 + \\ + a_{30} * x^3 + a_{21} * x^2 * y + a_{12} * x * y^2 + a_{03} * y^3$$

The confidence bounds on the coefficients determine their accuracy. In this study they are with 95% confidence bounds and are relevant to evaluate and compare fits. The coefficients of the surface fitting by a cubic polynomial in three dimensions are presented in Tab 3. Here x represents the values of ASP scoring function and it is normalized by mean 20.4, with the value of the standard distribution of the data 11.75; y represents the values of total energy and it is normalized by mean -0.3074, with the value of the standard

distribution of the data 297.9.The mean surface was calculated for all ligands and the polynomial coefficients of this fit are presented in Tab. 3.

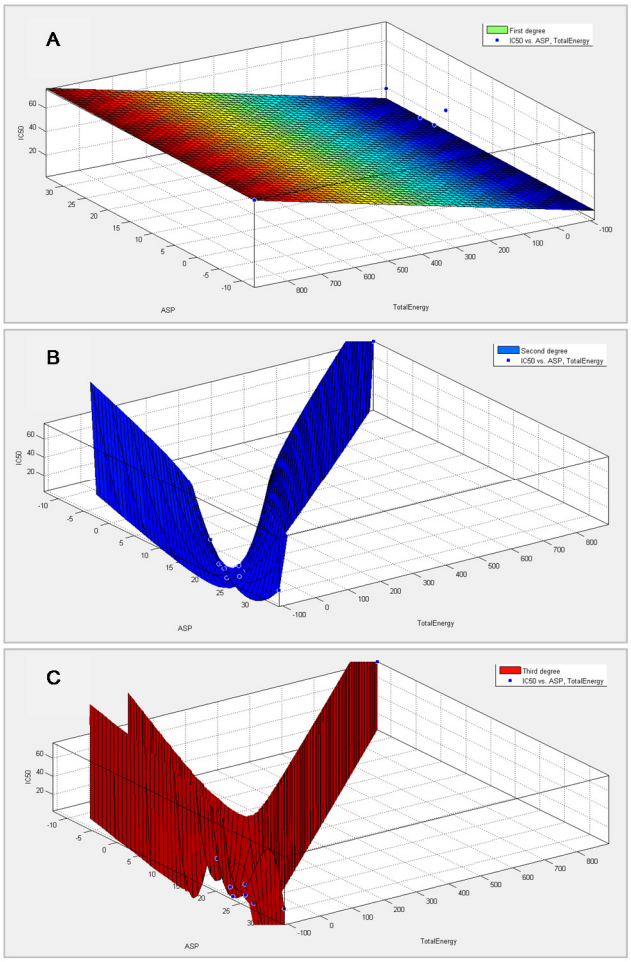

Fig. 2. A three-dimensional surface fitting of experimental data with polynomials of first to third degree, which representing the potency as a function of the values of ASP scoring function from docking procedure and the values of the total energy for ligand-receptor complex. The surface fitting with the first degree of polynomial model is presented in (A); with the second degree in (B) and the third degree in (C). The diagrams were generated with MATLAB.

The potency as a function of the values of ASP scoring function of docking experiments and the values of the total energy was presented in Fig. 2 (A,B and C) with a polynomial surface fitting of first to third degree in MATLAB. As it can be seen from the graphical representation of the experimental data (Tab.2) the best polynomial surface fitting is obtained for third degree of polynomial model (Fig.3 C).

Table 3: The mean values with confidence bounds of the coefficients of the third order polynomial model chosen as optimal.

Coefficients	Mean	95% confidence bounds
a_{00}	416.5	(319.1; 514)
a_{10}	-2420	(-3089; -1751)
a_{01}	111.7	(-248.9; 472.4)
a_{20}	-3299	(-3796; -2801)
a_{11}	-2.164	(-2.687; -1.639)
a_{02}	-1.439	(-1.829; -1.049)
a_{30}	-864.7	(-989.5; -739.8)
a_{21}	-1.301	(-1.493; -1.109)
a_{12}	-4.613	(-5.623; -3.602)
a_{03}	-3.382	(-4.211; -2.552)

A graphic representation of the relationship among three numeric variables in two dimensions is presented in Fig.4. The values of the ASP scoring function and the values of total energy are for X and Y axes, and the values of the potency are for contour levels. For the fitting by a cubic polynomial in 3D the contour plot (Fig.4) makes it easier to see points that have the same height. As it can be seen in Fig. 4 the best results are obtained for the polynomial model of third degree.

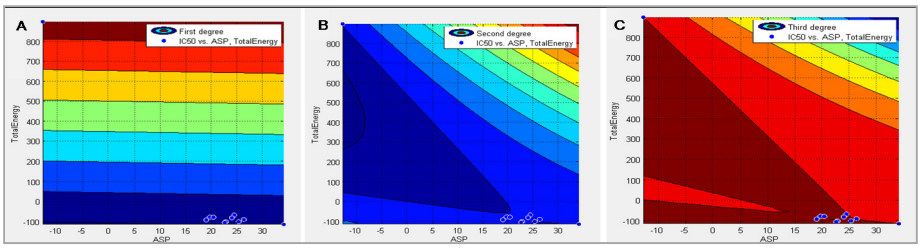

Fig.4: A 2D contour plot of 3D surface in the Fig.3. X represents the values of ASP scoring function and Y represents the values of total energy: (A) the first degree of polynomial fitting; (B) the second degree of polynomial fitting; (C) the third degree of the polynomial fitting. These diagrams were generated with the MATLAB.

The residual plot for obtained polynomial model is presented in Fig.5. It provides visual displays for assessing how well the model fits the data, for evaluating the distribution of the residuals, and for identifying influential observations. The top plot of residual diagram shows that the residuals are calculated as the vertical distance from the data point to the fitted curve. The bottom plot displays the residuals relative to the fit, which is the zero line. As it can be seen in Fig. 5 the obtained model is randomly scattered around zero. This indicates that the polynomial model of third degree describes the experimental data in a good way.

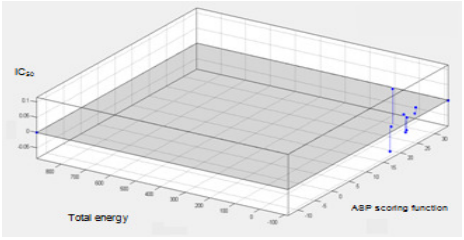

Figure 5: Residuals Plot for the obtained polynomial model.

Polynomial models are among the most frequently used empirical models for curve fitting and they are popular for the following reasons: they have a simple form, well known and understood properties, moderate flexibility of shapes and are computationally easy to use. The regression function is linear in terms of the unknown parameters. This allows easy to find the optimal regression coefficients using least squares method.

Therefore this model of third degree is suitable for presentation of the structure-activity relationship of delta selective enkephalin analogues.

4. ACKNOWLEDGMENTS

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Semi-Canonical Binary Matrices

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***Abstract**: In this paper, we define the concepts of semi-canonical and canonical binary matrix. Strictly mathematical, we prove the correctness of these definitions. We describe and we implement an algorithm for finding all $n \times n$ semi-canonical binary matrices taking into account the number of 1 in each of them. This problem relates to the combinatorial problem of finding all pairs of disjoint $n^2 \times n^2$ S-permutation matrices. In the described algorithm, the bitwise operations are substantially used.*

***Keywords:** binary matrix, ordered n-tuple, semi-canonical and canonical binary matrix, disjoint S-permutation matrices, bitwise operations*

MSC[2010]: 05B20, 68N15

1. INTRODUCTION

Binary (or *boolean*, or *(0,1)-matrix*) is called a matrix whose elements belong to the set $\mathcal{B} = \{0,1\}$.

Let n and m be positive integers. With $\mathcal{B}_{n \times m}$ we will denote the set of all $n \times m$ binary matrices, while with $\mathcal{B}_n = \mathcal{B}_{n \times n}$ we will denote the set of all square $n \times n$ binary matrices.

A square binary matrix is called a *permutation matrix*, if there is just one 1 in every row and every column. Let us denote by $\mathcal{P}_n$ the group of all $n \times n$ permutation matrices, and by $\mathcal{S}_n$ the symmetric group of order n, i.e. the group of all one-to-one mappings of the set $[n] = \{1, 2, \ldots n\}$ in itself. In effect is the isomorphism $\mathcal{P}_n \cong \mathcal{S}_n$.

As it is well known [4,5] the multiplication of an arbitrary real or complex matrix A from the left with a permutation matrix (if the multiplication is possible) leads to dislocation of the rows of the matrix A, while the multiplication of A from the right with a permutation matrix leads to the dislocation of the columns of A.

Let n be a positive integer and let $A \in \mathcal{B}_n$ be a $n^2 \times n^2$ binary matrix.

With the help of $n-1$ horizontal lines and $n-1$ vertical lines A has been separated into n^2 of number non-intersecting $n\times n$ square sub-matrices A_{kl}, $1\le k,l\le n$, e.i.

$$A=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}.$$

A matrix $A\in\mathcal{B}_{n^2}$ is called an *S-permutation* if in each row, each column, and each sub-matrice A_{kl}, $1\le k,l\le n$ of A there is exactly one 1. Two S-permutation matrices A and B will be called *disjoint*, if there are not $i,j\in\left[n^2\right]=\left\{1,2,\ldots,n^2\right\}$ such that for the elements $a_{ij}\in A$ and $b_{ij}\in B$ the condition $a_{ij}=b_{ij}=1$ is satisfied.

The concept of S-permutation matrix was introduced by Geir Dahl [1] in relation to the popular Sudoku puzzle. Obviously a square $n^2\times n^2$ matrix M with elements of $\left[n^2\right]=\{1,2,\ldots,n^2\}$ is a Sudoku matrix if and only if there are S-permutation matrices $A_1,A_2,\ldots,A_{n^2}$, each two of them are disjoint and such that M can be given in the following way:

$$M=1\cdot A_1+2\cdot A_2+\cdots+n^2\cdot A_{n^2}. \tag{1}$$

In [2] Roberto Fontana offers an algorithm which randomly gets a family of $n^2\times n^2$ mutually disjoint S-permutation matrices, where $n=2,3$. In $n=3$ he ran the algorithm 1000 times and found 105 different families of nine mutually disjoint S-permutation matrices. Then using (1) he obtained $9!\cdot 105=38\,102\,400$ Sudoku matrices.

Bipartite graph is the ordered triplet $g=\langle R_g,C_g,E_g\rangle$, where R_g and C_g are non-empty sets such that $R_g\cap C_g=\varnothing$, the elements of which will be called *vertices*. $E_g\subseteq R_g\times C_g=\{\langle r,c\rangle\mid r\in R_g,c\in C_g\}$ - the set of *edges*. Repeated edges are not allowed in our considerations.

If $x\in\{1,2,\ldots,n\}$, $\rho\in\mathcal{S}_n$, then the image of the element x in the mapping ρ we denote by $\rho(x)$. Let $g'=\langle R_{g'},C_{g'},E_{g'}\rangle$ and $g''=\langle R_{g''},C_{g''},E_{g''}\rangle$. We will say that the graphs g' and g'' are *isomorphic*

and we will write $g' \cong g''$, if $R_{g'} \cong R_{g''}$, $C_{g'} \cong C_{g''}$, $|R_{g'}| = |R_{g''}| = m$, $|C_{g'}| = |C_{g''}| = n$ and there exist $\rho \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$ such that $\langle r, c \rangle \in E_{g'} \Leftrightarrow \langle \rho(r), \sigma(c) \rangle \in E_{g''}$. In this paper we consider only bipartite graphs up to isomorphism.

Analyzing the works of G. Dahl [1] and R. Fontana [2], the question of finding a general formula for counting disjoint pairs of $n^2 \times n^2$ S-permutation matrices as a function of the integer n naturally arises. This is an interesting combinatorial problem that deserves its consideration. The work [7] solves this problem. To do that, the graph theory techniques have been used. It has been shown that to count the number of disjoint pairs of $n^2 \times n^2$ S-permutation matrices, it is sufficient to obtain some numerical characteristics of the set of all bipartite graphs considered to within isomorphism of the type $g = \langle R_g, C_g, E_g \rangle$, where $V = R_g \cup C_g$ is the set of vertices, and E_g is the set of edges of the graph g, $R_g \cap C_g = \varnothing$, $|R_g| = |C_g| = n$ $\left|E_g\right| = k$, $k = 0, 1, \ldots, n^2$.

Let $g = \langle R_g, C_g, E_g \rangle$ be a bipartite graph, where $R_g = \{r_1, r_2, \ldots, r_n\}$ and $C_g = \{c_1, c_2, \ldots, c_n\}$. Then we build the matrix $A = [a_{ij}] \in \mathcal{B}_n$, such that $a_{ij} = 1$ if and only if $\langle r_i, c_j \rangle \in E_g$. Inversely, let $A = [a_{ij}] \in \mathcal{B}_n$. We denote the i-th row of A with r_i, while the j-th column of A with c_j. Then we build the bipartite graph $g = \langle R_g, C_g, E_g \rangle$, where $R_g = \{r_1, r_2, \ldots, r_n\}$, $C_g = \{c_1, c_2, \ldots, c_n\}$ and there exists an edge from the vertex r_i to the vertex c_j if and only if $a_{ij} = 1$. It is easy to see that if g and h are two isomorphic graphs and A and B are the corresponding matrices, then A is obtained from B by a permutation of columns and/or rows.

Thus, the combinatorial problem to obtain and enumerate all of $n \times n$ binary matrices up to a permutation of columns or rows having exactly k units naturally arises. The present work is devoted to this problem.

2. SEMI-CANONICAL AND CANONICAL BINARY MATRICES

Definition 1. Let $A \in \mathcal{B}_{n \times m}$. With $r(A)$ we will denote the ordered n-tuple

$$r(A) = \langle x_1, x_2, \ldots, x_n \rangle,$$

where $0 \le x_i \le 2^m - 1,\ i = 1,2,\ldots n$ and x_i is a natural number written in binary notation with the help of the i-th row of A.

Similarly with $c(A)$ we will denote the ordered m-tuple

$$c(A) = \langle y_1, y_2, \ldots, y_m \rangle,$$

where $0 \le y_j \le 2^n - 1,\ \ j = 1,2,\ldots m$ and y_j is a natural number written in binary notation with the help of the j-th column of A.

We consider the sets:

$$\begin{aligned} \mathcal{R}_{n\times m} &= \left\{ \langle x_1, x_2, \ldots, x_n \rangle \mid 0 \le x_i \le 2^m - 1, i = 1,2,\ldots n \right\} \\ &= \left\{ r(A) \mid A \in \mathcal{B}_{n\times m} \right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_{n\times m} &= \left\{ \langle y_1, y_2, \ldots, y_m \rangle \mid 0 \le y_j \le 2^n - 1,\ j = 1,2,\ldots m \right\} \\ &= \left\{ c(A) \mid A \in \mathcal{B}_{n\times m} \right\} \end{aligned}$$

With "<" we will denote the lexicographic orders in $\mathcal{R}_{n\times m}$ and in $\mathcal{C}_{n\times m}$

It is easy to see that Definition 1 describes two mappings:

$$r : \mathcal{B}_{n\times m} \to \mathcal{R}_{n\times m}$$

and

$$c : \mathcal{B}_{n\times m} \to \mathcal{C}_{n\times m},$$

which are bijective and therefore

$$\mathcal{R}_{n\times m} \cong \mathcal{B}_{n\times m} \cong \mathcal{C}_{n\times m}.$$

Definition 2. Let $A \in \mathcal{B}_{n\times m}$,

$$r(A) = \langle x_1, x_2, \ldots, x_n \rangle,$$

$$c(A) = \langle y_1, y_2, \ldots, y_m \rangle.$$

We will call the matrix A *semi-canonical*, if

$$x_1 \le x_2 \le \cdots \le x_n$$

and

$$y_1 \le y_2 \le \cdots \le y_m.$$

Proposition 1. *Let* $A=[a_{ij}]\in\mathcal{B}_{n\times m}$ *be a semi-canonical matrix. Then there exist integers* i, j *, such that* $1\le i\le n$ *,* $1\le j\le m$ *and*

(2) $$a_{1\,1}=a_{1\,2}=\cdots=a_{1\,j}=0,\quad a_{1\,j+1}=a_{1\,j+2}=\cdots=a_{1\,m}=1,$$

(3) $$a_{1\,1}=a_{2\,1}=\cdots=a_{i\,1}=0,\quad a_{i+1\,1}=a_{i+2\,1}=\cdots=a_{n\,1}=1.$$

Proof. Let $r(A)=\langle x_1,x_2,\dots x_n\rangle$ and $c(A)=\langle y_1,y_2,\dots y_m\rangle$. We assume that there exist integers p and q, such that $1\le p<q\le m$, $a_{1p}=1$ and $a_{1q}=0$. In this case $y_p>y_q$, which contradicts the condition for semi-canonicity of the matrix A. We have proven (2). Similarly, we prove (3) as well.

□

Corollary 1. *Let* $A=[a_{ij}]\in\mathcal{B}_{n\times m}$ *be a semi-canonical matrix. Then there exist integers* s,t *, such that* $0\le s\le m$ *,* $0\le t\le n$ *,* $x_1=2^s-1$ *and* $y_1=2^t-1$

□

Definition 3. Let $A,B\in\mathcal{B}_{n\times m}$. We will say that the matrices A and B are equivalent and we will write

(4) $$A\sim B,$$

if there exist permutation matrices $X\in\mathcal{P}_n$ and $Y\in\mathcal{P}_m$, such that

(5) $$A=XBY.$$

In other words $A\sim B$ if A is received from B after dislocation of some of the rows and the columns of B.

Obviously, the introduced relation is an equivalence relation.

Definition 4. We will call the matrix $A\in\mathcal{B}_{n\times m}$ *canonical matrix*, if $r(A)$ is a minimal element about the lexicographic order in the set $\{r(B)\,|\,B\sim A\}$.

If the matrix $A\in\mathcal{B}_{n\times m}$ is canonical and $r(A)=\langle x_1,x_2,\dots,x_n\rangle$, then obviously

(6) $$x_1\le x_2\le\cdots\le x_n.$$

From definition 4 immediately follows that in every equivalence class about the relation "$\sim$" (definition 3) there exists only one canonical matrix.

Therefore, to find all bipartite graphs of type $g = \langle R_g, C_g, E_g \rangle$, where $V = R_g \cup C_g$ is the set of vertices, and E_g is the set of edges of the graph g, $R_g \cap C_g = \varnothing$, $| R_g | = | C_g | = n$, $\left| E_g \right| = k$, up to isomorphism, it suffices to find all canonical matrices with k 1's from the set $\mathcal{B}_{n\times n}$.

With $\mathcal{T}_n \subset \mathcal{P}_n$ we denote the set of all *transpositions* in $\mathcal{P}_n$, i.e. the set of all $n\times n$ permutation matrices, which multiplying from the left an arbitrary $n\times m$ matrix swaps the places of exactly two rows, while multiplying from the right an arbitrary $k\times n$ matrix swaps the places of exactly two columns.

Theorem 1. *Let A be an arbitrary matrix from $\mathcal{B}_{n\times m}$. Then:*

a) *If $X_1, X_2, \cdots, X_s \in \mathcal{T}_n$ are such that*

$$r(X_1X_2 \dots X_sA) < r(X_2X_3 \dots X_sA) < \cdots < r(X_sA) < r(A),$$

then

$$c(X_1X_2 \dots X_sA) < c(A).$$

b) If $Y_1, Y_2, \cdots, Y_t \in \mathcal{T}_m$ are such that

$$c(Y_1Y_2 \dots Y_tA) < c(Y_2Y_3 \dots Y_tA) < \cdots < c(X_tA) < r(A),$$

then

$$r(Y_1Y_2 \dots Y_tA) < r(A).$$

Proof. a) Induction by s. Let $s = 1$ and let $X \in \mathcal{T}_n$ be a transposition which multiplying an arbitrary matrix $A = [a_{ij}] \in \mathcal{B}_{n\times m}$ from the left swaps the places of the rows of A with numbers u and v $(1 \le u < v \le n)$, while the remaining rows stay in their places. In other words if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{u1} & a_{u2} & \cdots & a_{ur} & \cdots & a_{um} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{v1} & a_{v2} & \cdots & a_{vr} & \cdots & a_{vm} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} & \cdots & a_{nm} \end{bmatrix}$$

then

$$XA = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{v1} & a_{v2} & \cdots & a_{vr} & \cdots & a_{vm} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{u1} & a_{u2} & \cdots & a_{ur} & \cdots & a_{um} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} & \cdots & a_{nm} \end{bmatrix},$$

where $a_{ij} \in \{0,1\}$, $1 \le i \le n$, $1 \le j \le m$.

Let $r(A) = \langle x_1, x_2, \ldots, x_u, \ldots, x_v, \ldots, x_n \rangle$. Then $r(XA) = \langle x_1, x_2, \ldots, x_v, \ldots, x_u, \ldots, x_n \rangle$. Since $r(XA) < r(A)$, then according to the properties of the lexicographic order $x_v < x_u$. According to Definition 1 the representation of x_u and x_v in binary notation with an eventual addition if necessary with unessential zeros in the beginning is respectively as follows:

$$x_u = a_{u1}a_{u2}\cdots a_{um},$$

$$x_v = a_{v1}a_{v2}\cdots a_{vm}.$$

Since $x_v < x_u$, then there exists an integer $r \in \{1,2,\ldots,m\}$, such that $a_{uj} = a_{vj}$ when $j < r$, $a_{ur} = 1$ and $a_{vr} = 0$.

Hence if $c(A) = \langle y_1, y_2, \ldots, y_m \rangle$, $c(XA) = \langle z_1, z_2, \ldots, z_m \rangle$, then $y_j = z_j$ when $j < r$, while the representation of y_r and z_r in binary notation with an eventual addition if necessary with unessential zeroes in the beginning is respectively as follows:

$$y_r = a_{1r}a_{2r}\cdots a_{u-1r}a_{ur}\cdots a_{vr}\cdots a_{nr},$$

$$z_r = a_{1r}a_{2r}\cdots a_{u-1r}a_{vr}\cdots a_{ur}\cdots a_{nr}.$$

Since $a_{ur} = 1$, $a_{vr} = 0$, then $z_r < y_r$, whence it follows that $c(XA) < c(A)$.

We assume that for every s-tuple of transpositions $X_1, X_2, \ldots, X_s \in \mathcal{T}_n$ and for every matrix $A \in \mathcal{B}_{n\times m}$ from

$$r(X_1X_2\ldots X_sA) < r(X_2\cdots X_sA) < \cdots < r(X_sA) < r(A)$$

it follows that

$$c(X_1X_2\ldots X_sA) < c(A)$$

and let $X_{s+1} \in \mathcal{T}_n$ be such that

$$r(X_1X_2\dots X_sX_{s+1}A) < r(X_2\cdots X_{s+1}A) < \cdots < r(X_{s+1}A) < r(A).$$

According to the induction assumption $c(X_{s+1}A) < c(A)$.

We put

$$A_1 = X_{s+1}A.$$

According to the induction assumption from

$$r(X_1X_2\dots X_sA_1) < r(X_2\cdots X_sA_1) < \cdots < r(X_sA_1) < r(A_1)$$

it follows that

$$c(X_1X_2\cdots X_sX_{s+1}A) = c(X_1X_2\cdots X_sA_1) < c(A_1) = c(X_{s+1}A) < c(A),$$

with which we have proven a).

b) is proven similarly to a).

□

Obviously in effect is also the dual to Theorem 1 statement, in which everywhere instead of the sign "<" we put the sign ">".

Corollary 2. *If the matrix $A \in \mathcal{B}_{n\times m}$ is a canonical matrix, then it is a semi-canonical matrix.*

Proof. Let $A \in \mathcal{B}_{n\times m}$ be a canonical matrix and $r(A) = \langle x_1, x_2, \dots, x_n \rangle$. Then from (6) it follows that $x_1 \le x_2 \le \cdots \le x_n$. Let $c(A) = \langle y_1, y_2, \dots, y_m \rangle$. We assume that there are s and t such that $s \le t$ and $y_s > y_t$. Then we swap the columns of numbers s and t. Thus we obtain the matrix $A' \in \mathcal{B}_{n\times m}$, $A' \neq A$. Obviously $c(A') < c(A)$. From Theorem 1 it follows that $r(A') < r(A)$, which contradicts the minimality of $r(A)$.

□

In the next example, we will see that the opposite statement of Corollary 2 is not always true.

Example 1. We consider the matrices:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

After immediate verification, we find that $A \sim B$. Furthermore $r(A) = \langle 3,3,4,8 \rangle$, $c(A) = \langle 1,2,12,12 \rangle$, $r(B) = \langle 1,6,6,8 \rangle$, $c(B) = \langle 1,6,6,8 \rangle$. So A and B are two equivalent to each other semi-canonical matrices, but they are not canonical. Canonical matrix in this equivalence class is the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

where $r(C) = \langle 1,2,12,12 \rangle$, $c(C) = \langle 3,3,4,8 \rangle$.

From example 1 immediately follows that in a given equivalence class it is possible to exist more than one semi-canonical element.

3. ROGRAMME CODE OF AN ALGORITHM FOR FINDING ALL SEMI-CANONICAL MATRICES

Corollary 2 is useful that it is enough to seek canonical matrices from among the semi-canonical.

In this section, we are going to suggest an algorithm (Algorithm 1) for finding the semi-canonical matrices without checking all elements of the set $\mathcal{B}_{n\times m}$, described with the help of programming language C++. In the described algorithm, bitwise operations are substantially used. In [3] and [6] we prove that the representation of the elements of $\mathcal{B}_n$ using ordered n-tuples of natural numbers and bitwise operations leads to making a fast and saving memory algorithms. Similar techniques we used in the article [8], where we describe an algorithm for solving the combinatorial problem for finding the semi-canonical matrices in the set consisting of all $n\times n$ binary matrices having exactly k 1's in every row and every column. The results of this work are given in the Encyclopedia of Integer Sequences [10], respectively under the numbers A229161, A229162, A229163 and A229164. N. J. A. Sloane, who cites the work [9], presents all of them.

Algorithm 1. Receives all $n\times n$ semi-canonical binary matrices.

```
#define n …
/*
The function check(int[], int) verifies whether obtained n-tuple represents
a semi-canonical matrix and returns the number of 1's in the matrix.
*/
```

```
int check(int x[], int n)
{
    int k=0; //  The number of 1's in the matrix. If the matrix is not
                // semi-canonical, the function returns -1.
    int yj;     // The number represents the (n-j)-th column of the matrix
    int y0=-1; // The number before yi
    for (int j=n-1; j>=0; j--)
    {
        yj=0;
        for (int i=0; i<n; i++)
        {
            if (1<<j & x[i])
        {
          yj |= 1 << (n-1-i);
          k++;
        }
        }
        if (yj<y0 ) return -1;
        y0 = yj;
    }
    // This n-tuple represents a semi-canonical matrix. We print it.
    for (int i=0; i<n; i++) cout<<x[i]<<"  ";
    cout<<"k="<<k<<'\n';
    return k;
}

int main(int argc, char *argv[])
{
int x[n];  // x[n]-ordered n-tuple of integers that represent the rows
             // of the matrix
int k[n*n+1];  //k[i] - Number of semi-canonical matrices with
                  // exactly i 1's, 0<= i<=n*n
int m=n*n;
    for (int i=0; i<=m; i++) k[i]=0;
    int xmax=(1<<n)-1;
    int p,c;
    for (int s=0; s<=n; s++)
    {
      for (int i=0; i<n; i++)  x[i] = (1<<s)-1;
      c=check(x,n);
      k[s*n]++;
      p=n-1;
      while (p>0 && x[p]<xmax)
```

```
        {
            x[p]++;
            for (int i=p+1; i<n; i++)  x[i]=x[p];
            c = check(x,n);
            if (c>=0) k[c]++;
            p=n-1;
            while (x[p] == xmax)  p--;
        }
    }

    for (int i=0; i<=m; i++)
            cout<<"k("<<n<<","<<i<<") = "<<k[i]<<endl;
}
```

4. RESULTS

Let us denote with $\kappa(n,i)$ the number of all $n \times n$ semi-canonical binary matrices with exactly i 1's, where $0 \le i \le n^2$. Using Algorithm 1, we received the following integer sequences:

$$\{\kappa(2,i)\}_{i=0}^{4} = \{1,1,3,1,1\}$$

$$\{\kappa(3,i)\}_{i=0}^{9} = \{1,1,3,8,10,9,8,3,1,1\}$$

$$\{\kappa(4,i)\}_{i=0}^{16} = \{1,1,3,8,25,49,84,107,121,101,72,41,24,8,3,1,1\}$$

$$\{\kappa(5,i)\}_{i=0}^{25} = \{1,1,3,8,25,80,220,524,1057,1806,2671,3365,3680,3468,2865,$$
$$2072,1314,723,362,166,72,24,8,3,1,1\}$$

$$\{\kappa(6,i)\}_{i=0}^{36} = \{1,1,3,8,25,80,283,925,2839,7721,18590,39522,74677,125449,$$
$$188290,252954,305561,332402,326650,290171,233656,170704,113448,68677,$$
$$37996,19188,8910,3847,1588,613,299,72,24,8,3,1,1\}$$

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On the Classification of Boolean Functions

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***Abstract**: We define and discuss three complexity measures for Boolean functions and three equivalence relations which classify Boolean functions into classes with same number of implementations, subfunctions or separable sets. These relations induce three transformation groups. Using a class library for Boolean functions manipulation we have executed three algorithms to compute these new complexity measures.*

***Keywords:** subfunctions, implementations, separable sets.*

1. BASIC DEFINITIONS AND NOTATION

Understanding the complexity of Boolean functions is still one of the fundamental tasks in the theory of computation. In this section we introduce the basic definitions and notation of subfunctions, separable and distributive sets, implementations, ordered decomposition trees (ODTs) and ordered binary decision diagrams (OBDDs). Here we summarize basic results proved in [3] which we use in Section 2 to describe procedures for counting the complexity measures of Boolean functions depending on at most five variables.

An n-ary Boolean function is defined as a mapping: $f: \boldsymbol{B}^n \to \boldsymbol{B}$ where $\boldsymbol{B} = \{0,1\}$. P_2^n denotes the set of all n-ary Boolean functions. It is well known fact that there are 2^{2^n} functions in P_2^n .

When replacing some variables in a function f with constants the resulting functions are called **subfunctions** of f. We will denote this by $g = f\left(x_{i_1} = c_{i_1}, \dots, x_{i_m} = c_{i_m}\right)$ and $g \preccurlyeq f$. $Sub(f) = \{g | g \preccurlyeq f\}$, $sub(f) = |Sub(f)|$.

A variable x_i is **essential** in f if there are $a, b \in \boldsymbol{B}$, such that $f(x_i = a) \neq f(x_i = b)$. $Ess(f)$ denotes the set of all essential variables of f. $ess(f) = |Ess(f)|$.

A set M of essential variables is **separable** in f if there is a subfunction g, $g \preccurlyeq f$ such that $Ess(g) = M$. $Sep(f)$ denotes the set of all separable sets in f. $sep(f) = |Sep(f)|$.

For each $m = 0,1, \dots, n$, we denote by $sub_m(f)$ the number of subfunctions in f with m essential variables, i.e. $sub_m(f) = |\{g \in Sub(f) | ess(g) = m\}|$ and by $sep_m(f)$ the number of separable sets in f which consist of m essential variables, i.e. $sep_m(f) = |\{M \in Sep(f) \mid |M| = m\}|$.

The initial and more fundamental results concerning essential variables and separable sets were obtained in the work of K. Chimev, O. Lupanov, A. Salomaa, and others.

Let M and J be two non-empty sets of essential variables in the function f with $M \cap J = \emptyset$. The set J is called **distributive** of M in f, if for each $\boldsymbol{c} \in \boldsymbol{B}^{|J|}$ it holds $M \nsubseteq Ess(g)$, where $g \preccurlyeq_J^c f$ (see [3]) and J is minimal with respect to the order $\subseteq$. $Dis(M, f)$ denotes the family of the all distributive sets of M in f.

There are solid arguments that OBDDs are the most efficient method for representation of Boolean functions. We shall illustrate this by the following example.

Example 1.1 Let $f = x_1x_2 \oplus \overline{x_1}x_2$, where $\oplus$ denotes the operation addition modulo 2 and $\overline{\ }$ is the unary operation negation. Figure 1 shows the ODT(s) for f which essentially depends on all its three variables. The node at the top labeled f is the function node.

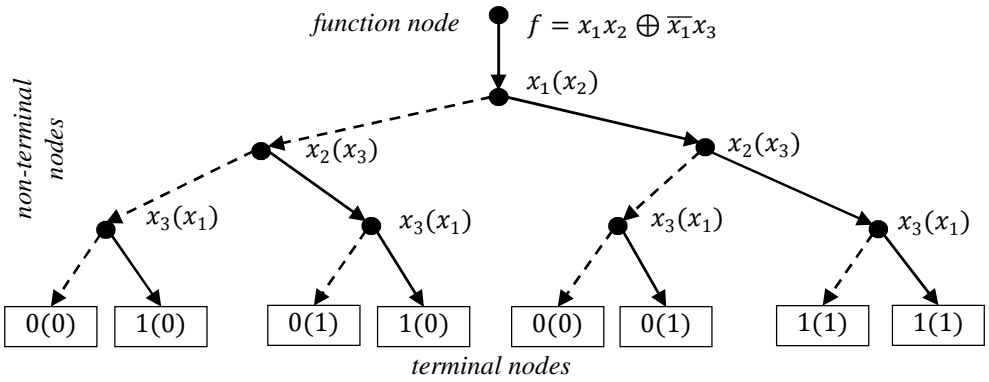

Fig. 1: ODT(s) for $f = x_1x_2 \oplus \overline{x_1}x_3$ under the orderings $\langle 1; 2; 3\rangle$ and $(\langle 2; 3; 1\rangle)$.

The nodes drawn as filled circles labeled with variable names are the **internal (non-terminal)** nodes, and the rectangular nodes (leaves of the tree) are the **terminal** nodes. The terminal nodes are labeled by the Boolean constants 0 or 1. Evaluation of f for a given valuation of x_1, x_2 and x_3 consists of selecting a path from the function node to a terminal node. The label of the terminal node is the value sought. At each non-terminal node the path follows the solid edge if the variable labelling the node evaluates to 1, and the dashed edge if the variable evaluates to 0.

The ordering in which the variables appear is the same along all paths of an ODT. It is known that for a given function f and a given ordering of its essential variables there is a unique ODT. For instance, when changing variable ordering $\langle 1;\ 2;\ 3\rangle$ with $\langle 2;\ 3; 1\rangle$ we obtain the same tree with labels drown in brackets (see Figure 1).

An **ordered binary decision diagram (OBDD)** [1] of a Boolean function is obtained from the corresponding ODT by **reduction** of its nodes applying

of the following two rules starting from the ODT and continuing until neither rule can be applied.

Reduction rules: (1) If two nodes are terminal and have the same label, or are non-terminal and have the same children, they are merged. **(2)** If a non-terminal node has identical children it is removed from the graph and its incoming edges are redirected to the child.

Figure 2 shows OBDDs of the function f under the variable orderings $\langle 1;\ 2;\ 3\rangle$ and $\langle 2;\ 3; 1\rangle$. Clearly the OBDD under $\langle 1;\ 2;\ 3\rangle$ is more simple than the diagram under $\langle 2;\ 3; 1\rangle$. This can be explained by the following observations: **(a)** the set $M = \{x_2, x_3\}$ is not separable in f; **(b)** the set $J = \{x_1\}$ is a distributive set of M.

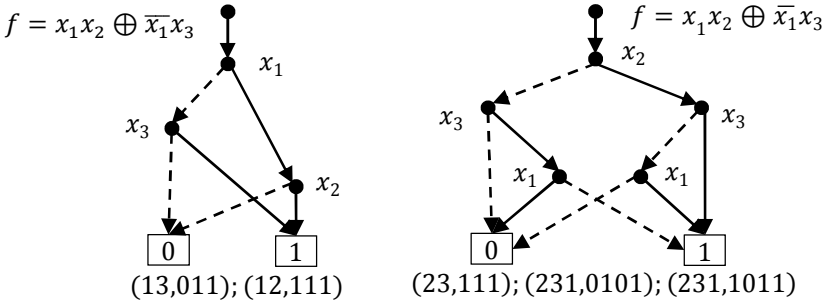

Fig. 2: OBDDs for $f = x_1x_2 \oplus \overline{x_1}x_3$ under the orderings $\langle 1; 2 ; 3\rangle$ and $(\langle 2; 3; 1\rangle)$.

Thus we conclude that the OBDDs which start with variables belonging to a distributive set of an inseparable set are simpler than others.

This example shows that the inseparable sets and their distributive sets play an important role for efficient representation of Boolean functions.

Each path in an OBDD of f, starting from the function node and finishing into a terminal node is called an **implementation** of the function f under the given variable ordering. The set of the all implementations of f, obtained under the all essential variable orderings, is denoted by $Imp(f)$. Each implementation of the function $f \in P_2^n$, obtained from the diagram D_f of f by the non-terminal nodes $x_{i_1}, \dots, x_{i_r}$ and corresponding constants $c_1, \dots, c_r, c \in \boldsymbol{B}$ with $f\left(x_{i_1} = c_1, \dots, x_{i_r} = c_r\right) = c, r \le ess(f)$, can be represented as a pair $(\boldsymbol{i}, \boldsymbol{c})$ of two words over $\boldsymbol{n} = \{1, \dots, n\}$ and $\boldsymbol{B}$ where $\boldsymbol{i} = i_1 i_2 \dots i_r \in \boldsymbol{n}^*$ and $\boldsymbol{c} = c_1 c_2 \dots c_r c \in \boldsymbol{B}^*$. Figure 2 shows also the implementations in presented OBDDs of f which finish in the terminal node labeled by 1.

The **implementation complexity** of a function $f \in P_2^n$ is defined as the number of all implementations of f, i.e. $imp(f) = |Imp(f)|$. Note that $imp(f)$ is determined by the function being represented but it does not depend on the chosen OBDD of the function.

The numbers $imp(f), sub(f)$ and $sep(f)$ characterize the computational complexity of the function f when calculating its values. Our next goal is to

classify the functions from P_2^n under these complexities. Using a class library for Boolean functions manipulation (BCL) explained in [4] calculation algorithms for this new complexity measures are implemented.

2. ALGORITHMS FOR BOOLEAN FUNCTIONS CLASSIFICATION

The functions from P_2^n can be represented by a 2^n bits binary number in the range from 0 to $2^{2^n}-1$ refering also as **function vector**. Techniques which involve enumeration of functions can only be used if n is trivially small. We have performed total exhaustion of the functions from P_2^n where $n < 6$. Subfunctions extraction is the main procedure in all our algorithms for calculation of complexity measures $imp(f)$, $sub(f)$ and $sep(f)$ executed with BCL. To extract all subfunctions of a given Boolean function when k variables are fixed we perform bitwise operations that reorder the bits in the function vector in such a way that each subfunction's bits form sequence of length 2^{n-k}.

Let each input state is represented by n-bit binary number i in the range $[0,2^n-1]$. Let the binary number m represents a bit mask of fixed variables. Then to find the new position p of the function value bit for the input state i we calculate $p = 2^{n-k}.P(m, m\&i) + P(\sim m, \sim m\&i)$, where & is bitwise logical operator and, $\sim$ is a bitwise negation operator, and $P(x,y)$ is a bitwise "packing" function. This function removes bits in y for which corresponding bits in x are 0, for instance $P(101001, 001001) = (000)011$.

Internally for BCL each Boolean function is represented as pair (v,s), where v is the function vector and s is the ordered list of function variables.

Theorem 2.1 [3] Let $f \in P_2^n$ be a Boolean function. Then $imp(f) = 1$ if $ess(f) = 0$ and $imp(f) = \sum_{x \in Ess\,(f)}\left[imp\big(f(x=0)\big) + imp\big(f(x=1)\big)\right]$ when f is not a constant.

We define the following three equivalence relations in P_2^n which classify functions with respect to their computational complexity.

Let $f, g \in P_2^n$ be two functions. Then $(i)\ \ f \simeq_{imp} g \iff imp(f) = imp(g)$; $(ii)\ f \simeq_{sub} g \iff sub_m(f) = sub_m(g)$; $(iii)\ f \simeq_{sep} g \iff sep_m(f) = sep_m(g)$ for all $m = 0, 1, \dots, n$.

These three equivalence relations induce three transformation groups IM_2^n, SB_2^n and SP_2^n, respectively, whose orbits are the equivalence classes under $\simeq_{imp}$, $\simeq_{sub}$ and $\simeq_{sep}$. In [3] we have proved that $(i)\ IM_k^n \le SP_k^n$; $(iii)\ IM_k^n \not\le SB_k^n$; $\ (ii)\ SB_k^n \le SP_k^n$; $\ (iv)\ SB_k^n \le IM_k^n$.

A mapping $\psi: P_k^n \to P_k^n$ is called **affine transformation** if $f = \psi(g)$ with: $f(\mathbf{x}) = g(\mathbf{xA} \oplus \mathbf{c}) \oplus \boldsymbol{a^t x} \oplus d$ where $\mathbf{x} = (x_1, \dots, x_n)$, $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$, $d, a_i, c_i \in \boldsymbol{B}$ and $\mathbf{A}$ is a non-singular binary matrix.

Many combinatorial results concerning the groups consisting of affine transformations are obtained in [2, 3]. We compare our groups with the lattice of all affine transformation groups. Let GE_2^n be the group of affine transformation when $\boldsymbol{A}$ is a permutation matrix and $\boldsymbol{a} = 0$ and let LG_2^n be the general linear group, i.e. the group of transformations when $\mathbf{A}$ is non-singular, $\mathbf{c} = \boldsymbol{a} = 0$ and $d = 0$. We have also proved in [9] that $(i)\ GE_k^n \leq IM_k^n$; $(ii)\ GE_k^n \leq SB_k^n$; $(iii)\ GE_k^n \leq SP_k^n$; $(iv)\ LG_k^n \not\leq SP_k^n$.

Hence each equivalence class under GE_2^n is included in a class under IM_2^n, in a class under SB_2^n, and in a class under SP_2^n. This result can be used to optimize the procedures described below.

The code below is used for counting the complexity $imp(f)$. It is based on an inductive method followed by Theorem 2.1.

Complexity Determined by Implementations

```
void ImpComplexity (int n) {
    BoolVector bv = BoolVector.Zero(n);
    do { BoolFunction bf = BoolFunction
                        .CreateFromVector(bv);
         var result = imp (bf);
        Print(bf, result);
    } while (BoolVector.Next(bv));
}
```

```
int imp (BooleanFunction bf) {
    if (bf.IsConstant()) return 1;
    if (bf.Ess().Count() == 1) return 2;
    else { int res = 0;
           foreach (var v in bf.Ess()) {
           BoolFunction sf0, sf1;
           bf.Subfunctions(v, out sf0, out sf1);
           res += imp (sf0) + imp (sf1);
       } } return res;
}
```

Complexity Determined by Subfunctions

The most time-consuming calculation is the calculation of complexity determined by subfunctions of a given Boolean function. This is because we need to compare functions sets rather numbers. The following C# like code outline the algorithm for calculation of $sub(f)$ complexity for all Boolean functions of n variables.

```
void SubComplexity (int n) {
    BoolVector bv = BoolVector.Zero(n);
    do { BoolFunction bf = BoolFunction.
                    CreateFromVector(bv);
        Dictionary<int, int> subf =
                     new Dictionary<int, int>();
        for (int i = 0; i <= bf.Variables().Count();
                                  i++) subf[i] = 0;
        List<BoolFuncton> list =
                  new List<BoolFuncton>();
        sub(bf.Reduce(), list);
        foreach(var f in list.Distinct())
                       subf[f.Ess().Count()]++;
        Print(bf, subf);
    } while (BooleanVector.Next(bv));
}
```

```
void sub(BoolFunction bf,
                List<BoolFunction> list) {
    list.Add(bf);
    if (bf.IsConstant) return;

    foreach (var v in bf.Variables()) {
        BoolFunction sf0, sf1;
        bf.Subfunctions(v, out sf0, out sf1);
        sub(sf0.Reduce(), list);
        if (sf0 ≠ sf1) sub(sf1.Reduce(), list);
    }
}
```

The next C# function outline our execution of counting procedure for separability complexity of all Boolean functions using BCL.

Complexity Determined by Separable Sets

```
void CalcSepComplexity () {
    BoolVector bv = BoolVector.Zero(n);
    do { BoolFunction bf = BoolFunction
                        .CreateFromVector(bv);
         Dictionary<int, int> sep =
                    new Dictionary<int, int>();
         for (int i = 1;
              i<=bf.Variables().Count();i++)
                  sep[i] = 0;
         foreach (var m in
                        bf.Variables().AllSets()) {
          if (bf.AreSeparable(m)) {
              sep[m.Length]++;
          }
        }
        Print(bf, sep);
    } while (BoolVector.Next(bv));
}
```

Table 1 shows the number of equivalence classes under the equivalence relations defined above. Note that there are $2^{32} = 4294967296$ functions in P_2^5. All calculations were performed on a computer with two Intel Xeon E5/2.3 GHz CPUs. The execution with total exhaustion took 244 hours, for $t(SP_2^5)$.

Tab. 1: Number of classes under symmetry type, $\simeq_{imp}$, $\simeq_{sub}$ and $\simeq_{sep}$.

n	$t(G_2^n)$	$t(IM_2^n)$	$t(SB_2^n)$	$t(SP_2^n)$
1	3	2	2	2
2	6	4	4	3
3	22	13	11	5
4	402	104	74	11
5	1 228 158	1606	< 1 228 158	38

3. ACKNOWLEDGEMENT

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Building Real-Time Web Applications with SignalR and NoSQL Databases

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***Abstract**: Nowadays real-time web applications are vastly becoming popular. They are widespread in a variety of industries, including: online betting, financial and banking services, online games and many others. The aim of the applications is: to analyze and process big missives of data received by various 3rd parties, store the data which is to be used later on and to only send/push the updates to connected clients (vs. all available clients). In this paper we demonstrate the collaboration between SignalR and NoSQL database for creating a model of a real-time web application matching the needs of the online betting industry.*

***Keywords:** SignalR, real-time web application, NoSQL, big data*

1. INTRODUCTION

Real-time applications are gaining popularity [5]. They have been used in the online betting industry for quite a while- especially within the live betting sections where every situation, point or goal affects the odds of the bookmakers. Microsoft presented SignalR library as a method for creating real-time web applications. These applications have the same goals:

1. Analyze / process the data received;
2. Store the data which is to be used later on;
3. Deliver real-time data / updates to clients.

SignalR allows rapid application development of a real-time web applications which satisfies some of the aforementioned goals including: receiving / analyzing / processing data and sending updates.

Modern web applications still need to store big missives of data received at all times. Real-time systems require even greater efforts and methods for handling data storage and retrieval. The options are:

1. Old-school relational databases;
2. NoSQL databases.

Nowadays all web applications from purely presentational web sites to enterprise information systems are using databases. Ever since the dawn of this technology in the 70s the relational databases have no alternatives and as such, they have been handling the business needs for decades. Internet

had become the utmost used business environment and the web applications must satisfy the following needs:

1. Constant huge increase in the number of users operating
2. Greater missives of data which have to be stored and processed
3. More frequent database queries which have to be executed

However we can now also register the flaws of the relational database, amongst them worth mentioning are: decreased speed for data manipulation upon increasing the volume, low scalability etc.

The web based business is constantly changing and is now searching for alternatives which would match its needs, such as: enhance the speed of the web applications regardless of the increased volume of information and number of users. As a response to these needs the non-relational data bases (NoSQL databases) emerged.

2. EVOLUTION OF DATA DELIVERY TECHNIQUES

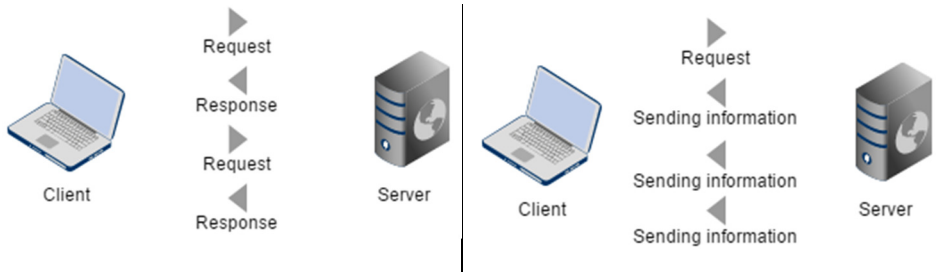
Recently tremendous effort has been invested in developing interactive, real-time multi-user systems. For building a real-time web applications could be applied different approaches: HTTP client pull, HTTP server push, web sockets, etc.

HTTP Client Pull: In the client pull technique, a new HTTP connection is opened every time a document is requested [4]. In a predefined period (Time to Refresh) the client calls the web server to receive updated data regardless of the presence (or the lack of) of any updates. The calls have to be frequent to sustain the information at a real-time pace. Data could be changed ten times for the next 10 seconds followed by a period of no updates in the next 2 minutes - yet again the client will continue to call the server with the same frequency – let's say every second. The main drawback is in the traffic generated by the client's calls and in keeping the server busy to respond regardless of the fact that there might be no updates. Amongst the versions of HTTP client pull could be found:

(i) **HTTP polling** (request → response) – client makes a request and waits for the server to respond [1]. If there is an update it is sent to the client. If there isn't one the server sends an empty response. Figure 1a) depict this communication mechanism.

(ii) **HTTP long polling** (request → wait → response) – Client creates a connection to the server, and it stays opened for a period of time [1]. Within this time slot the client can receive data from the server. The connection is closed right after the server sends its response. Client needs to reconnect after the connection is closed regardless if it's because of a time out or data being sent.

HTTP Server Push: In the server push technique, the connection is sustained until all data is delivered to the client [4]. The push is always initiated by the client. Server sends data to the clients periodically when there are updates. The biggest advantage is that the traffic is reduced tremendously. The communication between the server and the client occurs only when needed (updated information).



a) Request-Response b) Server Push

Fig. 1 Client Pull and Server Push communication models

HTTP Server push mechanism was further improved and standardized and in HTML5 it appear as so called **Server Sent Events** (SSEs). SSEs are sent over traditional HTTP. That means they do not require a special protocol or server implementation to get working. SSEs are handled directly by the browser and the user simply has to listen for messages.

The biggest disadvantage of the push mechanism is that communication is not bidirectional.

Web Sockets: The WebSocket Protocol for a bidirectional communication requires that both the client and the server applications are aware of the protocol details. This means you need a WebSocket-compliant Web page that calls into a WebSocket-compliant endpoint [2].

The client creates a TCP connection to the server which stays opened for as long as it needs. Client initiates the connection. Upon a successful attempt, the server and the client can exchange data in both directions. This is the main difference with Http Server push where the communication after the handshake is only from the server to the client. Using TCP protocol allows avoiding any performance issues.

Microsoft presents SignalR library as a way to simplify the process of adding real-time web functionality to applications. This is the ability to have server code pushing content to connected clients instantly as it becomes available, rather than having the server wait for a client to request new data [3]. SignalR makes possible the development of a brand new type of web

applications that require high frequency updates from the server such as online betting applications.

SignalR uses WebSocket where available, and falls back to older transports where necessary like HTTP long polling.

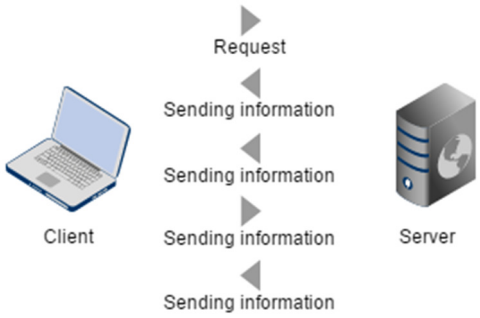


Fig. 2 Bidirectional communication via Web sockets

3. DATABASE HORIZONTAL SCALABILITY

There's a constant increase in terms of demands from the business. They are coming from the necessity to handle more and more users at all times, hence manipulating greater data missives. In order for the applications to meet these demands they must be scalable. The usage of a NoSQL implies **horizontal scalability** which would be applied in the online betting application. Horizontal scalability means adding new servers which are not that powerful and do not use expensive hardware. In general this approach is cheaper compared to a machine upgrade with the latest processors.

Pros: comes at a lower price, easier upgrading. Compared to a vertical scalability at a certain point both the further development and upgrade are no longer possible, i.e. the mother board cannot support further RAM enhancement or there's no better processor (in terms of productivity) available on the market.

Cons: more complicated implementation, necessity of a greater number of licenses, more room needed in the data center.

NoSQL are applicable for horizontal scalability. Approaches/Techniques for horizontal scalability:

Full Replication: In the majority of the cases there are several data base servers which own an identical copy of the data and are being synced. This is also called full replication. A single call is in theory handled by the DB server with the lowest load.

Sharding: Database sharding is a technique for achieving horizontal scalability in the applications. Sharding enables a single logic database to be transferred upon several servers, i.e. one could contain soccer odds, a

second server could contain tennis odds, followed by a different server for basketball odds and so on and so forth. This way the database requests are processed by different DB servers. This distribution of the data among several servers could result in data denormalization. The denormalization however would allow for separate fragments of the database to be independent hence a call to be handled by a single database shard [6].

4. CASE STUDY – ONLINE LIVEBETTING PLATFORM

The online betting industry is amongst the utmost aggressive ones – every point, goal or event during the game could change the bookies' odds. This requires not only a high performance but also a scalable application. To cover these goals we propose creating a SignalR hub which will process upcoming data and stores it in RavenDB shards for later on and also for analyses.

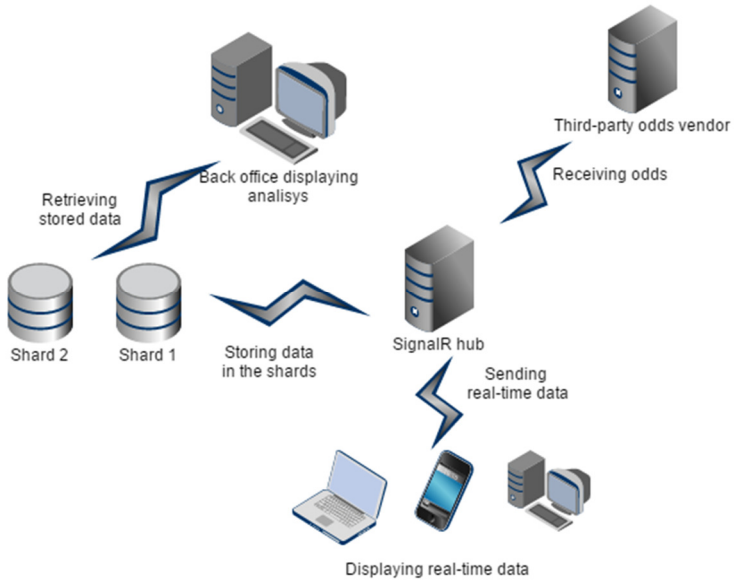


Fig. 3 Live-betting applications with SignalR hub and NoSQL database

Functionality of the application:

1. SignalR hub receives odds from third-party providers in the market.
2. The hub sends only updates or new data to the clients (browsers).
3. The hub also stores the data received in different RavenDB shards. The soccer odds are stored in the first shard, basketball ones in the second one and so on.
4. Pure HTML5 (HTML / CSS / KnockoutJS) page displays the live odds to the clients.

5. MVC back office displays the stored odds and different statistics to operators.

5. CONCLUSIONS

The model of the real-time web application we present in this paper is not the only available solution; however we do believe that it contains the following advantages:

- Using SignalR we gain performance – SignalR is based on TCP transport protocol which allows faster communication compared to other alternatives.
- Using SignalR we reduce the communication between the client and the server. Server sends only updates to the client if any. We skip the requests from the client to the server on a certain amount of time.
- Using SignalR we reduce the load of the server. The server uses the entire CPU resource to analyze and process the data received and sends to the clients only updates.
- Using SignalR we get native client support for desktop and mobile.
- Using NoSQL database we skip any performance problems which we would have had if we were to have a lot of data with the relational databases.

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Risk Analysis with R Language

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***Abstract**: This article is based on the latest techniques for measuring and managing on risks in various sectors of business. Using the programming language R Language we show effective way to evaluate and analysis risk.*

***Keywords:** R Language, risks, managing, risk analysis*

1. INTRODUCTION

Before proceeding to a discussion of the risks faced by a company before to analyze the state-of-the-art techniques to measure and manage these risks, we will start with the basic question in the management and analysis of financial risks: Why businesses need to manage risk?

Investors can eliminate specific risk by diversifying their companies to include many different assets. Some risks can be avoided through diversification, but exposure to these types of risk will be rewarded in the market. Instead, investors should hold a combination of non-risk assets and marketing portfolio, the exact combination depends on investors' appetite for risk. In this basic setup, companies should not waste resources, risk management, as investors are concerned about some specific risks, which assumes a company [1,2,3].

Not everyone chooses to manage the risk and the approach to risk management differ from one company to another. This is partly due to the fact that the objectives related to risk management vary across companies. In particular, some companies use the variability of cash flows, while others use the change in value of the company. It is also found that, overall, large companies tend to manage risk more active than smaller companies, which may be surprising, since small companies generally seen as more risky. However, smaller companies may have limited access to derivatives markets and also lack staff with skills in risk management.

Several studies in this area found that while the average prediction of risk tends to be too conservative, effectiveness over a period of time realized losses that exceed far risk estimates. Thus, the trends of excessive losses occur in consecutive days. Looking back at the data of each priority risk forecasts and former implementations we will be able to predict excessive loss for tomorrow based on observations of excessive loss today. This new

relationship reveals potential flaw in the current financial sectors in risk management and motivates the development and application of new tools [5,6].

2. MODELS OF INDIVIDUAL RISK

We will examine distribution function for a total of S benefits of portfolio insurance. To determine the cost of risk (Value-at-Risk - VaR), to tell the level of 99.5%, we need good approximations to the opposite of the cumulative distribution function (Cumulative Distribution Function - CDF).

Modeling no risks in insurance practices is a difficult task, as by pure discrete random variables, and also through continuous random variables. A simple and flexible model that produces random variables of this type is the mixed model, called " urn-of-urns".

Assuming risks in a portfolio there are independent random variables, the distribution of their sum can be calculated using convolution. Other alternative approaches that we can use to measure risk are momentary function generator (Moment Generating Function - MGF), characteristic function or probability generating function (Probability Generating Function - PGF) and cumulative generating function (Cumulant Generating Function - CGF) [4,16].

2.1. Mixed distributions and risks

Consider some examples of insurance risks. Let's have a claim by an insurance policy. First, expand small set of functions of distribution, considering purely discrete random variables and pure continuous random variables [7].

Of probability theory, we know that every function G(), which satisfies

(1) $G(-\infty)=0; G(+\infty)=1;$ G() is not decreasing and right continuous

is Cumulative Distribution Function (CDF) for a random variable. If G() is a step function, therefore it is a function which is constant beyond a countable set of discontinuities (steps), so G(), and each random variable X, for which G(x) = Pr[X ≤ x], is called discreet. The associated Probability Density Function (PDF) represents the height of the step for x so that

$$(2) \qquad g(x)=G(x)-G(x-0)=\Pr[X=x], \quad \forall x\in(-\infty,+\infty).$$

Here G(x - 0) stands for $\lim_{\varepsilon\to 0} G(x-\varepsilon): G(x+0)=G(x)$ and is retained because it is right-continuous. For each x, we have g(x) ≥ 0 and $\sum_x g(x)=1$, where the sum is taken over a countable set of all x for which g(x) > 0 [8,16].

Another special case is where G() is an absolute constant. This means that if g(x) = G'(x), then

$$(3) \qquad G(x) = \int_{-\infty}^{x} g(t)dt.$$

In this case, g() is also called probability density function.

The "urn-of-urns" model allows us to build a discrete random variable and continuous distribution. Let I be an indicator of a random variable with values I = 1 or I = 0, where I = 1 indicates that an event has happened. Suppose that the probability of the event is $q = \Pr[I = 1], 0 \le q \le 1.$ If I = 1, compensation Z is composed of the distribution of X, if I = 0 – the distribution of Y. This means that

$$(4) \qquad Z = IX + (1 - I)Y$$

If I = 1, then Z can be substituted with X, where I = 0 may be replaced with Y. We note that it may act not only when I, X and Y are independent, but in reality, and if the triple (X, Y, I) is also independent: only conditional distributions for X|I = 1 and Y|I = 0 are important. Therefore, it can record CDF for Z in the following way

$$(5) \quad G(\zeta) = \Pr[Z \le \zeta] = \Pr[Z \le \zeta, I = 1] + \Pr[Z \le \zeta, I = 0] = q \Pr[X \le \zeta] + (1 - q) \Pr[Y \le \zeta].$$

3. METHODOLOGY

Let's look at the case in which premium benefits are paid in a given year, as well as compensation for an upraised event still insurer is notified. In this case we have lost, to be recovered in future years. Such benefits are related to the years for which premiums have actually been paid. This means that we must maintain reserves with respect to known benefits that eventual size is unknown. For this purpose we can use several techniques associated with this type of benefits, such as benefits IBNR (Incurred But Not Reported - IBNR) claims that have occurred but have not registered. Hence the name IBNR methods, IBNR benefits and IBNR reserves for all benefits of this type [9,10].

In the statistical literature often mentions heuristics for the realization of IBNR technique. One of the most commonly used method for IBNR forecasts Chain Ladder method. There Summary Linear Model (Generalized Linear Model - GLM), in which some evaluators can be calculated with the method Chain ladder. On the other hand it is possible to obtain a model in which calculation method Chain pillars in terms of mean square error.

3.1. GLM which comprises IBNR methods

Commonly used statistical methods for the realization of IBNR technique can be described by a Generalized Linear Model (GLM). In Table 1, random variable X_{ij}, i, j = 1, 2, ..., t denote the amounts of claims with starting year i and development years j, which means that the debts were repaid during the calendar year i + j – 1.

Tab.1. Random variables in run-off triangle

Start year	1	⋯	Current year t-n+1	⋯	t
1	X_{11}	⋯	$X_{1,t-n+1}$	⋯	X_{1t}
⋮	⋮		⋮		⋮
n	X_{n1}	⋯	$X_{n,t-n+1}$	⋯	X_{nt}
⋮	⋮		⋮		⋮
t	X_{t1}	⋯	$X_{t,t-n+1}$	⋯	X_{tt}

Combinations (i, j), where i + j - 1 ≤ t, X_{ij} indicated, otherwise it is future monitoring. You can use these values to denote values, the ratio of loss. Using multiplicative model parameters for each row i, column j, and each diagonal k = i + j - 1, as follows:

$$X_{ij} \approx \alpha_i \cdot \beta_j \cdot \gamma_k \tag{6}$$

The deviation of the mean value for the right side is due to chance. Assuming that the random variables are independent X_{ij} and limiting their distribution in the exponential number of dispersions (7) is Generalized Linear Model. Initial year i, j year of development and the calendar year k = i + j - 1 act as explanatory variables for monitoring X_{ij}. The expected value of the exponent is X_{ij} linear form $\log\alpha_i + \log\beta_j + \log\gamma_k$, that we have a logarithmic relationship. Once you find the maximum probability estimates of parameters α_i, β_j and γ_k, under various assumptions about the probability distribution for X_{ij}, can easily expand IBNR triangle to a square:

$$\hat{X}_{ij} := \hat{\alpha}_i \cdot \hat{\beta}_j \cdot \hat{\gamma}_k \tag{7}$$

One of the problems is that we have data values γ_k for future calendar year k, where k > t. The problem can be solved as supposing there γ_k geometric model where $\gamma_k \propto \gamma^k$ for some real number γ [11,16].

3.1.1. De Wilder's Method of least squares

In the De Wilder's method of least squares (1978) we assume that $\gamma_k \equiv 1$ is withheld until α_i and β_j is determined by minimizing the sum of squares $\sum_{i,j}\left(X_{ij} - \alpha_i\beta_j\right)^2$ taken over the plurality (i, j), which are available

observations. This is tantamount to determining the α_i and β_j maximum likelihood model as follows:

$$(8) \qquad X_{ij} \approx N\left(\alpha_i \beta_j, \sigma^2\right) independant; \gamma_k \equiv 1.$$

De Wilder suggested using a multiplicative model $\alpha_i\beta_j\gamma_k$ for data X_{ij} and take values obtained for the parameters α_i, β_j and γ_k, which minimize the method of least squares. Thus solve

$$(9) \qquad \min_{\alpha_i \beta_j \gamma_k} \sum_{i,j} w_{ij} \left(X_{ij} - \alpha_i \beta_j \gamma_{i+j-1}\right)^2$$

If y_{ij} actual observation multiply it by weights $w_{ij} = 1$, otherwise referred $w_{ij} = 0$ so that it can accept the sum over all combinations (i, j). De Wilder proceed by making fixed inflationary component, therefore $\gamma_k = \gamma^k$ for some real γ and proves that doing so generally go beyond inflation model, thus take $\gamma_k \equiv 1$. Then he describes a method for consistent change to solve the problem associated with the reduction [12,16].

3.1.2. Weighted Historical Simulation (WHS)

Considered technique Historical Simulation (HS), describe relatively more weight on recent observations and relatively less weight to future returns of the past. This technique is called Weighted Historical Simulation (WHS) [13,14,15,16]. WHS is implemented as follows:

- Quote of m alleged past returns $\{R_{PF,t+1-\tau}\}_{\tau=1}^{m}$, determine the likely severity of exponential decline in the past as follows:

$$\mu_\tau = \left\{ \mu^{\tau-1} \frac{1-\mu}{1-\mu^m} \right\}_{\tau=1}^{m}$$

  so that, for example, today's observation will determine the weight $\mu^1 = (1 - \mu) / (1 - \mu^m)$). We note that μ_τ tends to zero when τ receive more value and therefore the sum of the weights μ_τ for $\tau = 1, 2, ..., m$ is 1. Typically μ adopt a number between 0.95 and 0.99.
- Observations along with their specific weights are sorted in ascending order.
- The value from 100p% for the variable VaR is calculated by accumulating the weights reached at 100p% upward return. Again, we can use linear interpolation to calculate the exact value of the VaR between the two sorted returns with cumulative probability weights around river.

We note that once chosen μ, WHS technique does not require assessment and thus retains the ease of implementation. Thus weighting function builds the dynamics in the art.

An obvious disadvantage of WHS approach is that it provides guidance on how to choose μ. Another disadvantage is the effect of the scheme for determining the weights of positive to negative past returns - the disadvantage is that WHS is part of the HS.

4. RESULTS

Suppose that we are interested in buying a portfolio of shares of Company A and Company B. We analyze historical data consisting of weekly prices of this share for three calendar years. Our goal is to predict possible future behavior of a portfolio consisting of a combination of the two types of shares.

Consider sets containing weekly share prices over three years for Company A and Company B. Figure 1 shows the behavior in time of stock prices by standard graphics.

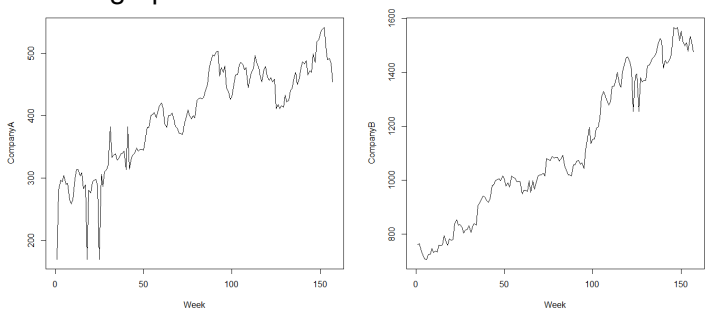

Fig. 1: Prices of shares of Company A and Company B

If S_t, t = 1, 2, ..., are stock prices, may look clean or arithmetic return $(S_{t+1} - S_t)/(S_t)$. In the condition of Black and Shoulz based on geometric Brownian motion, the logarithmic gain is independent and normally distributed. You can easily calculate the logarithmic returns (Fig. 2).

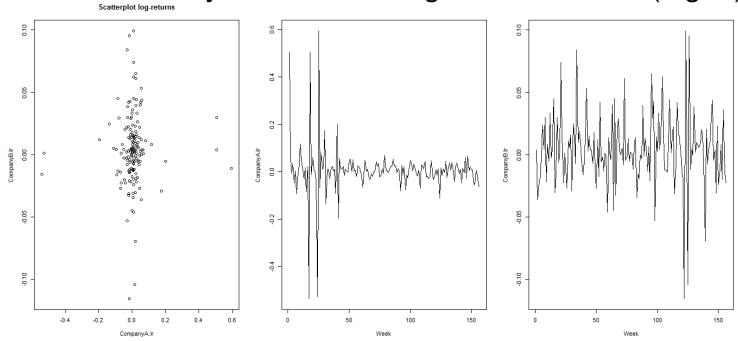

Fig. 2: Dot graphics and a time series graph of the logarithmic returns

To assess whether the marginal distributions are indeed normal, normal Q-Q checking plotted graphs (Fig. 3).

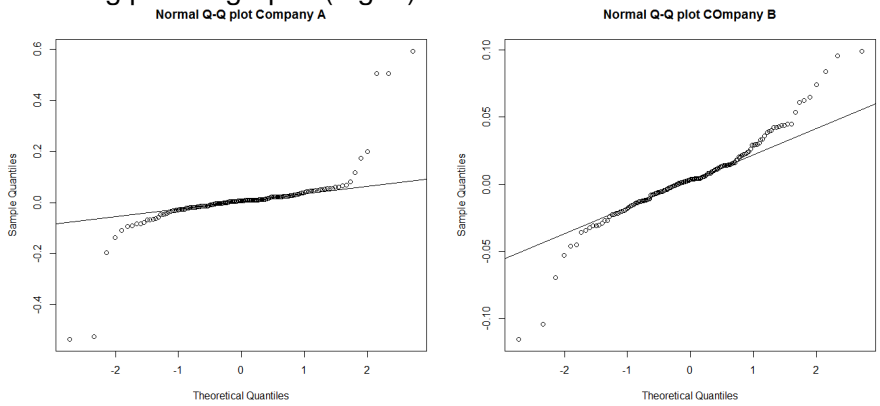

Fig. 3: Normal Q-Q plots of logarithmic returns

With Q-Q graphs plotted sample of quintiles compared to the theoretical quintiles. If normal Q-Q graphs are close to a straight line, probably limits are normal. In our case, the entry is not very good: the diagonal line passes through some points of comparison that is located about a quarter above or below the median and not closely monitored. Therefore, the sample of quintiles in queues is greater than in the case of normality.

Other graphical tool with which to assess normality is by plotting a histogram and comparing with appropriate normal density (Fig. 4).

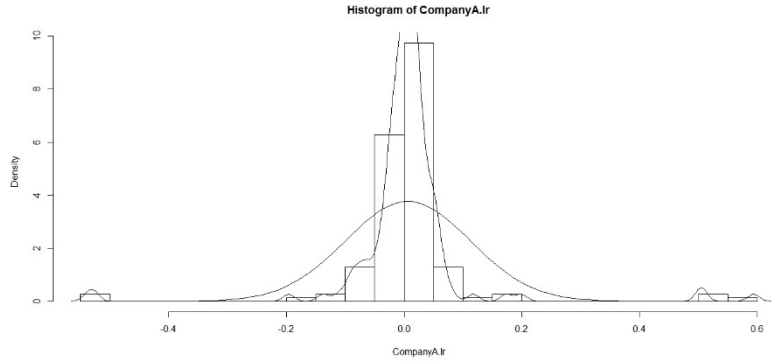

Fig. 4: Histogram of logarithmic returns

Logarithmic returns are not just normal limit, and in fact are bivariate normal. The chart in Figure 3 cannot understand anything about capital having joint normal distribution. But to see whether this assumption is valid can look scatter plots of logarithmic returns in Figure 2.

Suppose you want to buy a portfolio with equal parts of shares of Company A of the current price of 474 and shares of Company B on price

1498. interested in their future action over an interval of two calendar years or 104 weeks. Based on our data may assume that log weekly earnings (Xi, Yi), i = 158 ... 261 has a bivariate normal distribution. Therefore write

$$(10) \qquad X = X_{158} + \cdots + X_{261}; Y = Y_{158} + \cdots + Y_{261},$$

random quantity which must forecasting is

$$(11) \qquad S = 474 e^{X} + 1498 e^{Y}$$

The fact that $Cov(\sum X_i, \sum Y_i) = \sum Cov(X_i, Y_i)$, if couples (Xi, Yi) are independent, can calculate estimates of parameters (X, Y). Since X and Y are bivariate normal, S in Fig. 5 is the sum of the dependent logarithmic normal random variables. The calculation of the CDF and quantiles of S is difficult problem. One way of action is simply to simulate many results for the S model and seek instead theoretical quintiles for S. Figure 5 shows histograms to assess the density of the core and entered normal density.

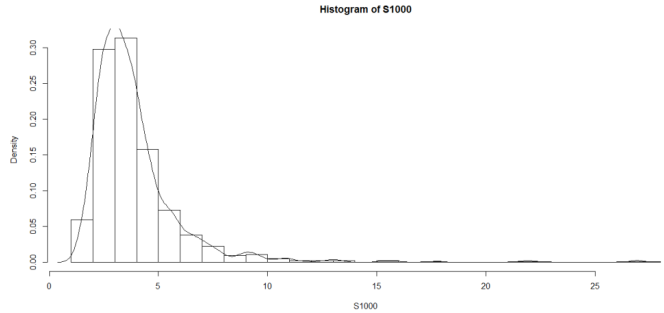

Fig. 5: Histogram of the S/100 with an estimate of the density of the core and a normal distribution

For a sample of quintiles for S, as a percentage of the purchase price, we get:

5%	10%	25%	50%	75%	90%	95%	97.5%	99%	99.5%
1711	1976	2325	3050	4106	5485	7233	8304	10150	12078

So, judging by quintiles 5%, 50% and 97.5%, every two years, the profits are approximately 50% ± 25%. Portfolio of one share of Company A and one share of Company B with the current price 474 + 1498 = 1972 after two years with a probability of 50% will be worth at least 152% more current value with 90% probability will be between 109% and 207%. This will reduce the amount of about one in forty cases. But one of the 200 cases, the portfolio will be sold for more than 2.5 times the initial value. All this is provided that future share prices will follow the expected pattern for our latest prices.

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A Publicly Available Cross-Platform Lemmatizer for Bulgarian

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Abstract: *Our dictionary-based lemmatizer for the Bulgarian language presented here is distributed as free software, publicly available to download and use under the GPL v3 license[1]. The presented software is written entirely in Java and is distributed as a GATE plugin. To our best knowledge, at the time of writing this article, there are not any other free lemmatization tools specifically targeting the Bulgarian language. The presented lemmatizer is a work in progress and currently yields an accuracy of about 95% in comparison to the manually annotated corpus BulTreeBank-Morph[2], which contains 273933 tokens [1].*

Keywords: *Bulgarian grammar, NLP, GATE*

1. INTRODUCTION

Lemmatization is a fundamental natural language processing (NLP) task which automates the process of word normalization. The correct identification of the normalized form of a word is of particular importance for several other NLP tasks and can improve the accuracy of tasks like information extraction and information retrieval. The implementation of a lemmatizer can be a challenging task, especially in cases of highly inflectional languages such as the Bulgarian language. For example, the Bulgarian adjective 'рядък' *râdăk* - 'rare') has 9 inflected word forms (Table 1).

Table 1. The inflected word forms of the adjective 'рядък' (*râdăk* - 'rare')

Cyrillic script	Latin scirpt (ISO 9)	Gender	Number	Article	Extended form
редкия	redkiâ	masculine	singular	definite	no
редкият	redkiât	masculine	singular	definite full	no
рядка	râdka	feminine	singular	indefinite	no
рядката	râdkata	feminine	singular	definite	no

[1] https://github.com/grigoriliev/BGLangTools/releases
[2] http://www.bultreebank.org/btbmorf/

рядко	râdko	neuter	singular	indefinite	no
рядкото	râdkoto	neuter	singular	definite	no
редки	redki	–	plural	indefinite	no
редките	redkite	–	plural	definite	no
редки	redki	masculine	singular	–	yes

2. IMPLEMENTATION

The BGLemmatizer package is part of the BGLangTools project. For the implementation of the BGLemmatizer, we decided to reuse freely available lexical resources. We focused on two projects - BG Office[1] and Bulgarian-language Wiktionary[2]. We implemented support in BGLangTools for automatic retrieval of all lemmas provided by these projects and automatic word form generation for the lemmas having inflected forms.

The word forms of lexemes are generated by means of inflectional paradigms. In the Bulgarian language there are 187 inflectional paradigms (types), which cover almost all words with inflected forms [2]. To inflect all word forms of a lemma, we only need to know the inflectional paradigm to which the lemma belongs. In a nutshell, we need to know the lemma type. For example, to retrieve a list of all word forms of the Bulgarian adjective 'рядък' (râdăk - 'rare'), which is of type "83", we will use the following code:

```
WordEntry[] forms;
forms = BgWordFormGenerator.generateWordForms("рядък", "83");
```

Since BG Office and Bulgarian-language Wiktionary projects are work in progress and it is expected that the provided lexical resources will grow in time, BGLangTools provides an option to scan the projects data and import the newly added lemmas in its database. Currently, the dictionary contains 65 376 lemmas and 1 017 595 inflected forms. The number of inflected forms will grow, because at the time this paper was written the inflection of verbs has been partially implemented - only the most frequently used word forms could be generated. Note that in the Bulgarian language a verb can have up to 2000 word forms [3].

Most of the currently available part-of-speech (POS) taggers for Bulgarian use the BulTreeBank tagset (BTB-TS) scheme [4] to annotate words with POS tags. For efficiency reasons, in BGLangTools the grammatical information about a word entry is stored in a 32-bit integer, but

[1] http://bgoffice.sourceforge.net
[2] http://bg.wiktionary.org

for interoperability purposes, we implemented support for converting a BTB-TS POS tag to a 32-bit integer value and vice versa.

2.1. Retrieving a list of lemmas from the BG Office project

The goal of the BG Office project is to add spell check support for Bulgarian in various open source projects like OpenOffice (openoffice.org) and Mozilla (mozilla.org). BG Office provides software packages for developers[1], which contain lexical resources for Bulgarian, where the lemmas are sorted according to inflectional type and stored in separate files.

In the latest version (4.1) of the developers package the lexical resources are located in the `'data'` directory. In this directory and its sub-directories, there are several files with `.dat` extension whose names start with 'bg', followed by a number and an optional one-letter suffix. The number with the optional suffix designates the inflectional type of the lemmas contained in the file.

BGLangTools provides support for retrieval of all lemmas from a BG Office developers package. For every lemma the corresponding inflectional forms are generated and added to the dictionary. This can be done by means of the following code:

```
String bgofficePath = "/packages/bgoffice-4.1/";
BgDictionary dict = new BgDictionary();
BgOfficeScanner.getInstance().scan(bgofficePath, dict);
```

The `bgofficePath` variable specifies the absolute path to the top level directory of the BG Office package.

2.2. Retrieving a list of lemmas from the Bulgarian-language Wiktionary project

The goal of the Wiktionary project is to provide a complete international dictionary under a free license. The backup dumps of the content are available for free download[2].

The support for processing the Bulgarian-language Wiktionary dump file[3] is implemented in BGLangTools by using the Bliki engine[4]. The following code can be used to process the specified dump file, retrieve all lemmas, generate the corresponding inflectional forms and import them to the specified dictionary:

[1] http://sourceforge.net/projects/bgoffice/files/For%20Developers/
[2] http://dumps.wikimedia.org/
[3] http://dumps.wikimedia.org/bgwiktionary/latest/bgwiktionary-latest-pages-articles.xml.bz2
[4] https://code.google.com/p/gwtwiki/

```
String dump;
dump = "/dumps/bgwiktionary-latest-pages-articles.xml.bz2";
BgDictionary dict = new BgDictionary();
BgWiktionaryScanner.getInstance().scan(dump, dict);
```

The `dump` variable specifies the absolute path to the Wiktionary dump file.

2.3. *Using the built-in dictionary*

At the time this article was written, BGLangTools kept in internal format 1 082 971 tokens, retrieved from the BG Office project and the Bulgarian-language Wiktionary project. In order to load the internally stored tokens with their grammatical information, one can use the following code:

```
BgDictionary dict = BGLangTools.loadBuiltinDictionary();
```

3. USING THE BGLEMMATIZER PLUGIN FOR GATE

GATE (General Architecture for Text Engineering) is a mature language processing framework capable of handling various text processing tasks[1]. It is an open source project with over 15 years of development. We implemented support for running BGLemmatizer as GATE plugin.

Part-of-speech information is required for every token so that the BGLemmatizer plugin can work properly. This information can be automatically obtained by POS taggers like the LingPipe POS tagger, which is also available as GATE plugin. An example configuration of the BGLemmatizer plugin is shown in Figure 1, where the plugin is configured to retrieve the POS tag of a token from the 'category' feature of the 'Token' annotation and to set the resulting lemma as a 'lemma' feature in the same annotation.

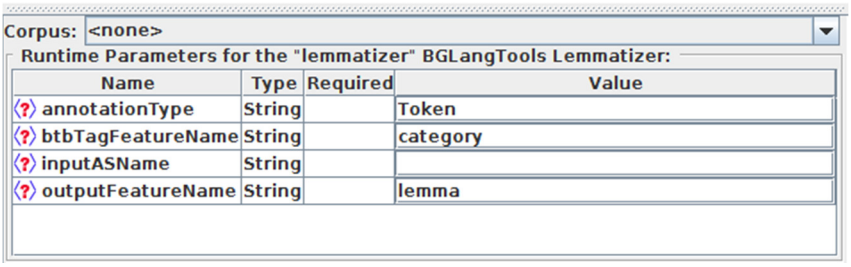

Figure 1. Runtime parameters of the BGLemmatizer plugin in GATE

[1] http://gate.ac.uk/

4. FUTURE WORK

We plan to integrate the BGLemmatizer with other linguistic software tools. One of our aims is to achieve better results and accuracy of part-of-speech annotation of Bulgarian texts with high levels of grammatical errors, which is the case with texts written by people learning Bulgarian as a foreign language. Further, we plan to extend BGLangTools with APIs, which will be useful in the fields of automatic text generation and information extraction.

5. ACKNOWLEDGEMENTS

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Evaluation of the Accuracy of the BGLemmatizer

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Abstract: *This paper reveals the results of an analysis of the accuracy of developed software for automatic lemmatization for the Bulgarian language. This lemmatization software is written entirely in Java and is distributed as a GATE plugin. Certain statistical methods are used to define the accuracy of this software. The results of the analysis show 95% lemmatization accuracy.*

Keywords: *Bulgarian grammar, NLP, GATE*

1. INTRODUCTION

The entry for "lemmatize" in the *Collins English Dictionary* states that lemmatization is the process of grouping together the different inflected forms of a word so that they can be analyzed as a single term (Collins English Dictionary[1]). Lemmatization is a fundamental natural language processing (NLP) task which automates the process of word normalization. The correct identification of the normalized form of a word is of significance for NLP tasks such as information extraction and information retrieval. It becomes of particular interest when applied to highly inflectional languages such as Bulgarian, which is a South Slavic language.

The total amount of relevant information in a sample is controlled by two factors:

- The sampling plan or experimental design which represents the procedure for collecting the information;
- The sample size *n* or the amount of information one collects [4].

The corpus we have investigated consists of 273933 tokens [1]. Every token is manually annotated with part-of-speech tags [5]. The aim of our study is to evaluate the lemmatizer's performance regarding three parts of speech, namely the noun, the adjective and the verb. In order to improve the accuracy of analysis results we have investigated only these parts of speech as a population with size 149061. We have conducted the survey in two stages (i.e. we have examined a two-phase sampling). Our discussion begins with an analysis of the results of a pilot study as such an approach has two

[1] http://www.collinsdictionary.com

advantages. First, the pilot study will be used to provide estimates of the individual stratum variances and second, the results of the pilot study can be used to estimate the number of observations needed to obtain estimators of the population parameters with a specified level of precision.

On the basis of the results obtained from the samples, the traditional evaluation metrics have been applied, namely, **Precision, Recall** and **F-measure** [2].

Precision, as is well-known, measures the number of the items that have been correctly identified as a percentage of the number of the identified items. The higher the **Precision** is, the better the system is at ensuring what has been identified as being correct.

Recall measures the number of the items that have been correctly identified as a percentage of the total number of items. The higher the **Recall** rate is, the better the system is at not missing correct items.

The **F_β-measure** is used together with **Precision** and **Recall,** as a weighted average of both **Precision** and **Recall**.

2. SURVEY

As mentioned above, we have conducted a pilot study, sampling 40 observations from each district, whereby the numbers in the three stratum district are [4]:

N_1=80509 /for the nouns/

N_2=23159 /for the adjectives/

N_3=45393 /for the verbs/

The numbers n_1, n_2, n_3 have been sampled in the three strata as follows:

$$(1) \quad n_j = \frac{N_j \sigma_j}{\sum_{i=1}^{3} N_i \sigma_i} \cdot n \qquad (j=1,2,3),$$

where σ_i marks the stratum population standard deviations. The sample stratum standard deviations obtained are σ_1=0,243, σ_2=0,352, σ_3=0,195.

By substituting our sample estimates in place of quantities (1), we have found out that:

n_1=0,534.n; n_2=0,224.n; n_3=0,242.n

We have now specified the proportions of the total sample to be allocated to each stratum under the optimal scheme.

By means of (2) we can find the total number of the sample:

$$(2) \quad n = \frac{\frac{1}{N}\left(\sum_{i=1}^{3} N_j \sigma_j\right)^2}{N\sigma_{\hat{p}}^2 + \frac{1}{N}\sum_{j=1}^{K} N_j \sigma_j^2}$$

where N=149061 is the total number of the population members and $\sigma_{\hat{p}}^2$ is the variance of the estimator of the population proportion.

In order that the 95% confidence interval for the population proportion be achieved we extend the error margin by 0,02 on each side of the sample estimate ($\sigma_{\hat{p}}$ =0,02).

Hence, we can conclude that the needed total number of sample observations is 597.

Given that it is easy to make a random sample, the total number of sample observations amounted to 1373. These have then been allocated among the three strata as follows:

n_1=0,534.1373=732
n_2=0,224.1373=308
n_3=0,242.1373=333

Since 40 observations have already been sampled in each stratum, the numbers sampled in the second phase are 692, 268, 293 respectively.

We have estimated the sample proportion by means of (3):

$$(3) \quad \widehat{p}=\frac{\sum_1^3 n_i . p_i}{\sum_1^3 n_i},$$

where p_i marks the proportions of the investigated parameters in each stratum and n_i marks the numbers of the sampled stratum [3].

In view of **Precision** we resave $\hat{P} = 0{,}97$ and 95% confidence interval for the population **Precision** is:

0,97 – 0,02 < P < 0,97 + 0,02

Concerning **Recall** we resave $\widehat{\mathrm{R}} = 0{,}93$ and *95%* confidence interval for the population **Recall** is:

0,93 – 0,02 < R < 0,93 + 0,02

In our study the **F-measure** is used as a weighted average of both **Precision** and **Recall** which are considered as equally important, so that:

$$F = \frac{P.R}{0{,}5(P+R)} = 0{,}95$$

Such results are highly satisfactory and bear out the high accuracy and precision of the developed lemmatizater.

For the purpose of the following analyses, we have designed the frequency distribution on the basis of the specific features of each part of speech.

In order to achieve greater objectivity of verification of the sample observations, the parts of speech are retrieved with context. The specifics of our study design facilitate the procedure of eliciting extensive information on the structure of the different parts of speech included in the corpus. In view of

the above, we have built the necessary frequency distributions which are demonstrated in the following tables.

Table 1. Frequency distribution of the Nouns

BTB-TS tag	Frequency
N-msi	18667
N-msh	4918
N-msf	2560
N-mpi	5004
N-mpd	3136
N-mt	1966
N-fsi	15816
N-fsd	6127
N-fpi	4992
N-fpd	1836
N-nsi	7398
N-nsd	4288
N-npi	1992
N-npd	986

Table 2. Frequency distribution of the Adjectives

BTB-TS tag	Frequency
Amsi	3256
Amsh	2062
Amsf	1099
Afsi	3287
Afsd	2785
Ansi	2074
Ansd	1492
A-pi	4172
A-pd	2811

Table 3. Frequency distribution of the Verbs

BTB-TS tag	Frequency	BTB-TS tag	Frequency	BTB-TS tag	Frequency
V---f-r1s	1769	V---u-o2p	20	V---cv—sfi	669
V---f-r2s	656	V---u-o3p	22	V---cv—sfd	130
V---f-r3s	15582	V---z--2s	217	V---cv—sni	522
V---f-r1p	1391	V---z---p	158	V---cv—snd	92
V---f-r2p	634	V---cao-smi	1204	V---cv--p-i	1305
V---f-r3p	6711	V---cao-smh	55	V---cv--p-d	327
V---f-t1s	42	V---cao-smf	3	V---car-smi	74
V---f-t2s	4	V---cao-sfi	478	V---car-smh	46
V---f-t3s	831	V---cao-sfd	120	V---car-smf	17
V---f-t1p	11	V---cao-sni	314	V---car-sfi	63
V---f-t2p	7	V---cao-snd	25	V---car-sfd	107
V---f-t3p	259	V---cao-p-i	941	V---car-sni	50
V---u-o1s	53	V---cao-p-d	133	V---car-snd	48
V---u-o2s	3	V---cv—smi	1124	V---car-p-i	165
V---u-o3s	112	V---cv—smh	96	V---car-p-d	162
V---u-o1p	5	V---cv—smf	50		

3. CONCLUSION

The above frequency distribution tables can be used in performing analysis of the errors and detecting the types of errors in view of their elimination which will contribute to increasing the precision and efficiency of the developed software and enhance the possibilities for its application in different NLP tasks, such as information extraction and information retrieval.

We suggest that the samples should be made randomly in keeping with the proportions presented in the frequency distributions. The proposed method of choosing the sample size can be used in performing the estimation procedure for the three cases listed above. The parameters that are to be estimated as well as the standard error margin are determined on the basis of their point estimator.

4. ACKNOWLEDGEMENTS

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A Review on Applications of Graph Theory in Network Analysis

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Abstract : *For many centuries ideas now embodied in graph theory have been implicit in lay discussions of networks. The explicit linking of graph theory and network analysis began only in 1953 and has been rediscovered many times since social network (and relationship network in general) are abundant and useful source of information. We can use graph theory to model them. However they can be difficult to analyse . We can learn more about them by calculating metrics and analyzing their statistics. Real graph frequently display power law degree distributions and small-world phenomena. Analysts have taken from graph theory mainly concepts and termmology; Its theorems, though potentially valuable for the analysis of real data, are generally neglected. Network analysts thus make too little USC of the theory of graphs. Some concept of graph theory for network analysis are noted. This work addresses the problem by presenting analysis of different researches on shortest path problem in various areas of applications.*

Key words: *graphs, graph theory, application of graphs, graph algorithms, etc.*

1. INTRODUCTION

Internet is the fastest medium which is now made accessible to every people from all the part of the world. Communication through internet is more specified, with effective interactive strategy among its users. In recent days, internet advertising has taken new forms which have more advantages over the traditional mediums. Social network analysis and graph theory provide a conceptual framework to study contact patterns and to identify units of analysis that are frequently or intensely connected within the network. Networks are used to move people,transport goods, communicate information and control the flow of matter and energy. Networks are all around us because of the frequent occurrence of such network structured problems, there is a need to develop an efficient procedure for handling these problems.

2. GRAPHS: MAIN DEFINITIONS

The mathematical models of network structures have been developed in graph theory. A network of elements (such as people, computers, firms, or roads) is usually represented by a diagram consisting of a number of dots (nodes or vertices) and a number of lines (arcs or edges) connecting certain pairs of dots. From a mathematical point of view, in such diagrams the important thing is whether or not two given points are connected by a line, and the nature of the connection is disregarded. This abstraction is what we call a ***graph*.**

Conceptually, *a graph* is formed by vertices and edges connecting the vertices

Formally,*a graph G* is an ordered pair of disjointed sets *(V, E)*, where *V = {v1,…vn}* is the set of vertices and *E = {(v_1,u_1),…, (vi,uj)}* is the set of arcs. *E* is a subset of the Cartesian product *V*×*V*. In other words, *E* is a binary relation on *V* and its elements are pairs of elements belonging to *V*.

Undirected Graph: A graph in which each edge symbolizes an unordered, transitive relationship between two nodes. Such edges are rendered as plain lines or arcs.

Directed Graph/Digraph: A graph in which each edge symbolizes an ordered, non-transitive relationship between two nodes. Such edges are rendered with an arrowhead at one end of a line or arc.

A walk in the graph G = (V, E) is a finite sequence of the form $v_{i_0}, e_{j_i}, v_{i_1}, e_{j_2},\ldots..e_{j_k}, v_{i_k}$, which consists of alternating vertices and edges of G

Path: A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.

A complete graph *Kn* of order *n* is a simple graph with *n* vertices in which every vertex is connected (adjacent) to every other. It has *n(n-1)/2* edges (all possible choices of pairs of vertices).

If the set of vertices *V* of a graph *G* can be divided into two disjointed non-null sets *A* e *B*, such that each edge of *G* connects a vertex in *A* with a vertex in *B*, the graph *G* is called *bipartite* (there are no 'internal' edges among the vertices in *A* or *B*.

The edge's *density* is the ratio between the actual number of links and the maximum possible number:

$$\delta = \frac{m}{\frac{1}{2}n(n-1)}$$

The number of edges connected to a vertex *u* is called vertex *degree deg*(*u*) A ***tree*** is a connected acyclic . In a weighted graph, the length *d(u,v)* of a path *p* is the sum of the weight *w* on the relevant edges:

$$d(p) = \sum w(v_{i-1}, v_i)$$

One more important parameter which describes a graph is its *characteristic (*or *average) path length L(G).* It is the arithmetic mean of the distance values calculated by considering all possible pairs of vertices. Since the number of pairs is *½(n(n-1)*, *L(G)* is defined by*:*

$$L(G) = \frac{2}{n(n-1)} \sum_{\{u,v\}} d(u,v)$$

A *connected* graph is an undirected graph that has paths between any two vertices. *Connected components* of a graph are the equivalence classes of vertices defined by the relation *reachable from*. A directed graph is *strongly connected* if it has a path from each vertex to every other vertex.

The maximum possible length for a path between any two vertices in a connected graph is *n-1* (*n* order of the graph); the number of cycles is: *l = m – n + 1*. If a graph contains no closed cycles (it is a *tree*) and is connected, then: *m = n – 1*.

3. ALGORITHMS AND GRAPH THEORY

The major role of graph theory in computer applications is the development of graph algorithms. Numerous algorithms are used to solve problems that are modeled in the form of graphs. These algorithms are used to solve the graph theoretical concepts which intern used to solve the corresponding computer science application problems. Some algorithms are as follows: 1. Shortest path algorithm in a network 2. Finding a minimum spanning tree 3. Finding graph planarity 4. Algorithms to find adjacency matrices. 5. Algorithms to find the connectedness 6. Algorithms to find the cycles in a graph 7. Algorithms for searching an element in a data structure (DFS, BFS) and so on. Various computer languages are used to support the graph theory concepts. The main goal of such languages is to enable the user to formulate operations on graphs in a compact and natural manner. Some graph theoretic languages are:

1. SPANTREE – To find a spanning tree in the given graph.
2. GTPL – Graph Theoretic Language
3. GASP – Graph Algorithm Software Package
4. HINT – Extension of LISP
5. GRASPE – Extension of LISP
6. IGTS – Extension of FORTRAN
7. GEA – Graphic Extended ALGOL (Extension of ALGOL)
8. AMBIT – To manipulate digraphs Language

4. A BRIEF HISTORICAL SKETCH OF GRAPH AND NETWORK THEORIES

The researcher has made an attempt to review the literature related to finding shortest path in various applications. However there are number of literatures, the researcher managed to review few articles / papers published in the journal and books.

List of other study title in journals & articles related to research problem:

1.The paper that fixes the start is *Solutio problematis ad geometriam situs pertinentis* written in 1736 by the Swiss mathematician Leonhard Euler. In it, Euler proposes a mathematical formulation of the renowned Königsberg Bridge Problem:

Is it possible to plan a walk through the town of Königsberg which crosses each of the town's seven bridges once and only once?

The puzzle problem, proposed to Euler by the Königsberg inhabitants, is stated in this way (the English translation of the original Latin paper can be found in Biggs et al., 1976): The importance of Euler's paper for the history of mathematics does not lie, obviously, in the solution of the game.

2. G*eometria situs*, as Leibniz had called it is today known with the name of topology, and Euler's solution is the first of this kind formally stated and solved. Despite the numerous but sparse works on this topic in the second part of the 18th and in the 19th centuries (Cauchy, Kirchoff, Hamilton, Poincaré, to quote just the most famous authors), a formal setting of these theories came only exactly 200 years after the Königsberg bridges paper. In 1936, the German mathematician Dénes König (1884-1944) published in Leipzig the first systematic study of what he called *graphs* in his *Theorie der endlichen und unendlichen Graphen* (König, 1936).

In the same period, ideas and techniques developed for the study of these abstract objects were applied to a completely different field. Realising that a group of individuals can be represented by enumerating the *actors* of the group and their mutual relationships, sociologists started using graph theory and methods to describe and analyse the patterns of social relations (Freeman, 2004; Wasserman & Faust, 1994). Jacob L. Moreno (1934) introduced *sociometry*. By using a *sociogram* (a diagram of points and lines used to represent relations among persons) he aimed at identifying the structure of relationships around a person, group, or organisation in order to study how these configurations may affect beliefs or behaviours.

Different lines of research stem from this origin. As Scott et al. (2000: 7) states: A number of very diverse strands have shaped the development of present-day social network analysis. These strands have intersected with one another in a complex and fascinating history, sometimes fusing and other times diverging on to their separate paths. A clear lineage for the mainstream of social network analysis can, nevertheless, be constructed from this complex history. In this lineage there are three main traditions: the

sociometric analysts, who worked on small groups and produced many technical advances with the methods of graph theory; the Harvard researchers of the 1930s, who explored patterns of interpersonal relations and the formation of 'cliques'; and the Manchester anthropologists, who built on both of these strands to investigate the structure of 'community' relations in tribal and village societies. These traditions were eventually brought together in the 1960s and 1970s, again at Harvard, when contemporary social network analysis was forged.

Common terms such as *weak ties* in a social context, or the *smallness* of the world of our acquaintances, come directly from seminal works in this area such as that by Mark Granovetter (1973) or the Stanley Milgram (1967) experiments.

The next major breakthrough in the history of network theories is a series of three papers published in the early 1960s by the Hungarian mathematicians Paul ErdYs and Alfréd Rényi (1959; 1960; 1961) on *random graphs*. The problem addressed was a fundamental question in the quest for understanding graphs, networks and interconnection phenomena: how do these objects form? and how do they evolve over time?

The approach used is statistical and probabilistic. The ErdYs-Rényi (ER) model has, since then, become a standard model, able to explain many of the characteristics of the networks encountered in the real world (see section 2.4.1). For almost 30 years the ER model, the only available model of this kind, has been used, investigated and developed by many authors.

In the last years of the 1990s, the Internet revolution had a tremendous impact on almost all aspects of our life. One of the crucial influences the Internet has had is in the birth of a completely new approach to the studies of networks. For the first time a huge mass of data was available to researchers. Data were easily accessible and usable for sophisticated statistical analyses, and the Internet was the primary source, or the medium, for this availability.

The beginning of the *'New' Science of Networks,* as it has been named (Watts, 2004), lies in three papers written in 1998-1999:

- *Collective dynamics of 'small world' networks*, by Watts and Strogatz (1998);
- *On power-law relationships of the internet topology* by M. Faloutsos, P. Faloutsos, and C. Faloutsos (1999); and
- *Emergence of scaling in random networks* by Barabási, and Albert (1999).

These works have provided evidence that the ER model was simply a crude approximation of only a special class of networks, and that many of those found in the real world, technological, physical, biological or social, exhibited characteristics and properties of a different nature. Since then, a

vast amount of work has been carried out and numerous phenomena have received explanation and have been modelled. Furthermore, it has strongly reinforced the idea that the collective properties of dynamic systems composed of a large number of interconnected parts are strongly influenced by the topology of the underlying network (see the bulky reviews by Albert & Barabási, 2002; Boccaletti et al., 2006; Dorogovtsev & Mendes, 2002; Newman, 2003b).

Lili Cao, Xiaohan Zhao, Haitao Zheng, and Ben Y. Zhao[2] conclude that search for shortest paths is an essential primitive for a variety of graph-based applications, particularly those on online social networks. They propose Atlas, a novel approach to scalable approximate shortest paths between graph nodes using a collection of spanning trees. They demonstrate its scalability and effectiveness using 6 large social graphs from Facebook, Orkut and Renren, the largest of which includes 43 million nodes and 1 billion edges. They describe techniques to incrementally update Atlas as social graphs change over time. They capture graph dynamics using 35 daily snapshots of a Facebook network, and show that Atlas can amortize the cost of tree updates over time. Finally, they apply Atlas to several graph applications, and show that they produce results that closely approximate ideal results.

Sanchit Goyal [3] studied Travelling Salesman Problem (TSP) problem in combinatorial optimizationin both, operations research and theoretical computer science. Given a list of cities and their pair wise distances, the task is to find a shortest possible tour that visits each city exactly once. It was first formulated as a mathematical problem in 1930 and is one of the most intensively studied problems in optimization

Shalu Wadhwa *(*2000) [5] in this work researcher's have targeted a Network Design Problem (Cable and Trench Problem), which involves a trade-off between utilization costs and capital costs for network construction. A larger network,(the shortest path tree) may cost more to build but may reduce utilization costs by including more attractive origin-destination paths. Conversely, a smaller network, (minimum spanning tree) may increase the utilization costs.

Kamal A. Ahmat [8] studied extensively in association with complex communication networks. They described basic concepts of graph theory and their relation to communication networks. Then they are presented some optimization problems that are related to routing protocols and network monitoring and showed that many of the optimization problems are NP-Complete or NP-Hard. Finally, they explained some of the common tools used to generate network topologies based on graph theory.

Yanghua Xiao [9] takes a problem of online answering shortest path queries by exploiting rich symmetry in graphs. The most famous and widely used algorithm to solve the shortest path problem is Dijkstra, which is fast

using heap data structures for priority queues shortest path queries are important in many applications.

Karsten M. Borgwardt and Hans-Peter Kriegel (2005) [12] defined graph kernels based on shortest paths, which are polynomial to compute, positive definite and retain expressivity while avoiding the phenomenon of ”tottering”. In experiments on classifying graphs model of proteins into functional classes, they outperformed kernels based on random walks significantly. The shortest-path kernels prevent tottering. It is not possible that the same edge appears twice in the same shortest path, as this would violate the definition of a path. Subsequently, artificially high similarity scores caused by repeated visiting of the same cycle of nodes are prohibited in our graph kernel. The shortest-path kernel as described in this article is applicable to all graphs on which Floyd-Warshall can be performed. Floyd-Warshall requires that cycles with negative weight do not exist. If edge labels represent distances, which is the case in most molecular classification tasks, this condition generally holds.

Je Chen, Jacob Steinhardt (2006) [10] concludes that, Dijkstra's Algorithm traversal algorithms are specialized for finding the shortest paths between vertices on the graph.

S.G.Shirinivas, S.Vetrivel and Dr. N.M.Elango (2010) [13] presented the importance of graph theoretical ideas in various areas of computer applications like Shortest path algorithm in a network, Finding a minimum spanning tree, Finding graph planarity, Algorithms to find adjacency matrices, Algorithms to find the connectedness, Algorithms to find the cycles in a graph, Algorithms for searching an element in a data structure

Sahar Abbasi and Sadoullah Ebrahimnejad (2011) [14] in this paper they considered the dynamic shortest path problem, motivated by its applications in dynamic minimum cost flows in transformation problem. They showed that this problem is equivalent to a classical shortest path problem in a so-called time-expanded network. Although our approach allows us to apply any standard technique on the time-expanded network, the size of this network is typically very large for realistic problems and it may be beneficial to avoid such explicit expansion. They used the Label Correcting Algorithm for solving this problem that the time complexity of the algorithm is O (|nT || mT |).

One more aspect of this work is also worth noting: the contributions to this *new science* are, probably for the first time in the history of science, truly and absolutely interdisciplinary. Physicists, mathematicians, computer scientists, biologists, economists, and sociologists are all equally contributing to the growth of the knowledge in this field.

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Measurement of Physical Values with Embedded Computer Systems

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***Abstract**: A embedded computer systems is a facility, either on land, with instruments and equipment for observing atmospheric conditions to provide information for weather forecasts and to study the weather and climate. The measurements taken include temperature, barometric pressure, humidity, wind speed, wind direction and etc. In this study are present results from such facility installed at South-West University of Blagoevgrad, Bulgaria. The basic platform is Raspberry Pi*

***Keywords:** Measurement of physical values, embedded computer systems, temperature;*

1. INTRODUCTION

The Raspberry Pi is a series of credit card-sized single-board computers developed in the UK by the Raspberry Pi Foundation with the intention of promoting the teaching of basic computer science in schools. The original Raspberry Pi and Raspberry Pi 2 are manufactured in several board configurations through licensed manufacturing agreements with Newark element14 (Premier Farnell), RS Components and Egoman.

The original Raspberry Pi is based on the Broadcom BCM2835 system on a chip (SoC), which includes an ARM1176JZF-S 700 MHz processor, VideoCore IV GPU,[7] and was originally shipped with 256 megabytes of RAM, later upgraded (models B and B+) to 512 MB.The system has Secure Digital (SD) (models A and B) or MicroSD (models A+ and B+) sockets for boot media and persistent storage. In 2014, the Raspberry Pi Foundation launched the Compute Module, which packages a BCM2835 with 512 MB RAM and an eMMC flash chip into a module for use as a part of embedded systems.

The Foundation provides Debian and Arch Linux ARM distributions for download. Tools are available for Python as the main programming language, with support for BBC BASIC (via the RISC OS image or the Brandy Basic clone for Linux), C, C++, Java, Perl and Ruby.

In early 2015, the next-generation Raspberry Pi, Raspberry Pi 2, was officially announced.The new computer board will initially be available only in one configuration (model B) and features a Broadcom BCM2836 SoC, with a quad-core ARM Cortex-A7 CPU and a VideoCore IV dual-core GPU; 1 GB of RAM with remaining specifications being similar to those of the previous generation model B+.

The Raspberry Pi primarily uses Linux-kernel-based operating systems. The ARM11 chip at the heart of the Pi (pre-Pi 2) is based on version 6 of the ARM. The current releases of several popular versions of Linux, including Ubuntu, will not run on the ARM11. It is not possible to run Windows on the original Raspberry Pi,though the new Raspberry Pi 2 will be able to run Windows 10. The Raspberry Pi 2 currently only supports Ubuntu Snappy Core, Raspbian, OpenELEC and RISC OS.

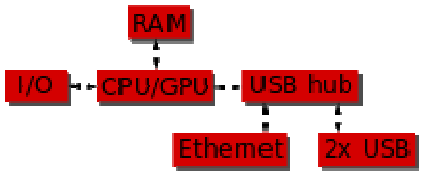

Fig.1. Architecture of Raspberry Pi

In the above block diagram for model A, B, A+, B+; model A and A+ have the lowest two blocks and the rightmost block missing (note that these three blocks are in a chip that actually contains a three-port USB hub, with a USB Ethernet adapter connected to one of its ports). In model A and A+ the USB port is connected directly to the SoC. On model B+ the chip contains a five-point hub, with four USB ports fed out, instead of the two on model B. [1]

2. METHODS

The Raspberry Pi is a fully functional, small and power efficient computer. It is ideal for the realization of measurement and control tasks in the home. The article describes the construction of a temperature measuring station based on the Raspberry Pi and the 1-wire temperature sensor DS1820. Thanks to the extensive software that is available for the operating system Raspbian the graphical presentation of the measured values in the network by a web server is easy to set up.

The temperature is measured with multiple sensors of the type DS18S20, DS18B20 or DS1822. The different types are pin and software compatible, they differ basically in the achievable measurement accuracy and price. The integrated circuit in a TO-92 case includes the temperature sensor, an analog

to digital converter, and a 1-wire interface. It is possible to drive the circuit with a parasitic power supply, which allows the use of the single terminal DQ for both data transmission and power supply. Hence the term "1-wire" comes. In addition to this "one wire" is required a second reference potential (ground, GND). One can therefore connect the DS1820 with a simple two-pole twisted pair cable.

In addition, the DS1820 still have a dedicated terminal VDD for the operating voltage of 3 to 5 volts. But this would require a three-core cable. Such active power is necessary if one wants to measure reliably temperatures above about 70° Celsius. VDD and GND are to be connected when using the parasitic power supply.

Multiple DS1820 may be operated parallel on a single 1-wire bus without additional components. Each sensor has a branded unique code for identification.

Temperature measurement with the Raspberry Pi and the 1-wire temperature sensor DS1820 contains a list of different ways to drive a 1-wire bus from the Raspberry Pi. This paper describes the solution (1) with the least amount of external circuitry. You need apart from the sensors only a single resistor, as this solution completely simulates the 1-wire protocol in software. [2]

The data port DQ of the DS1820 is connected directly to the port GPIO4 of the GPIO interface of the Raspberry Pi. GND and VDD are at ground terminal GND. The parasitic power supply accomplished a pull-up resistor of 4k7 between the 3.3 volt connection 3V3 and GPIO4.

The resistor, a female connector for attachment to the GPIO port and the 1-wire connection is soldered to a small breadboard, which sits directly on the GPIO port. The 1-wire connection can be realized with a space-saving angled female connector. All this fits into the TEK-BERRY housing of the Raspberry Pi.

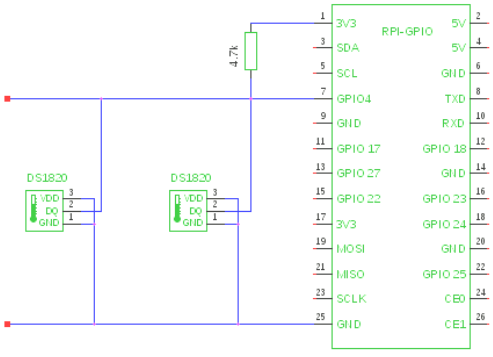

Fig.2. Connecting the DS18020 to the GPIO port of the Raspberry Pi

The kernel modules required for the activation of the 1-wire temperature sensor are part of the Linux distribution Raspbian "wheezy". You should, however, explicitly load them:

```
sudo modprobe w1-gpio pullup=1
sudo modprobe w1-therm
```

Important is the parameter pullup = 1, which tells the module that a parasitic power supply via a pull-up resistor is present. The modules create a subdirectory for each sensor found just below /sys/bus/w1/devices. The directory name is composed of the Family Code of the sensor and its unique identification number. Sensors of the type DS1820 and DS18S20 have the Family Code 10, DS18B20 has Code 28 and DS1822 the 22. In each subdirectory there is the file w1_slave containing the sensor status and measured temperature value:

```
cd /sys/bus/w1/devices
cd 10-000801b5*
cat w1_slave
0f 00 4b 46 ff ff 06 10 0c : crc=0c YES
0f 00 4b 46 ff ff 06 10 0c t=7375
```

The file consists of two lines, each containing the hexadecimal register-dump of the sensor IC. At the end of the first line is the checksum (CRC) and the information whether it is a valid reading (YES). The second line ends with the temperature reading in thousandths of a degree Celsius. In the example, the temperature is thus 7.375 °C. The accuracy to three places after the decimal point is of course only apparent; the datasheet of DS18S20 states, for example, that the measurement accuracy is only ± 0.5° C. The actual temperature is so anywhere from 6.8 to 7.9° C.

If everything works so far, you should enter the two required modules into the file /etc/modules to make them automatically loading at boot time:

```
# /etc/modules
w1-gpio pullup=1
w1-therm
```

A Python scripts reads the special files w1_slave and inserts the temperature values into the round-robin database:

```
#!/usr/bin/python
# -*- coding: utf-8 -*-
import re, os, rrdtool, time
# function: read and parse sensor data file
```

```
   def read_sensor(path):
     value = "U"
     try:
       f = open(path, "r")
       line = f.readline()
       if re.match(r"([0-9a-f]{2} ){9}: crc=[0-9a-f]{2}
YES", line):
         line = f.readline()
         m  =  re.match(r"([0-9a-f]{2}  ){9}t=([+-]?[0-
9]+)", line)
         if m:
           value = str(float(m.group(2)) / 1000.0)
       f.close()
     except (IOError), e:
       print  time.strftime("%x  %X"),  "Error  reading",
path, ": ", e
     return value

   # define pathes to 1-wire sensor data
   pathes = (
     "/sys/bus/w1/devices/10-000801b5a7a6/w1_slave",
     "/sys/bus/w1/devices/10-000801b5959d/w1_slave"
   )
   # read sensor data
   data = 'N'
   for path in pathes:
     data += ':'
     data += read_sensor(path)
     time.sleep(1)
   # insert data into round-robin-database
   rrdtool.update(
     "%s/temperature.rrd"                                %
(os.path.dirname(os.path.abspath(__file__))),
     data)
```

The reading of w1_slave is made in the function read_sensor. It tests whether the first line ends with YES and thus there exists a valid checksum. If so, then the function extracts the temperature value from the second line of the file and returns the value in degrees Celsius. In case of error, it returns the value U, which is interpreted as "Unknown" by RRDtool.

In the main program are first the definitions of the paths to the temperature sensors - these of course you need to change! Then the script reads in all of the sensors by calling the function read_sensor. The waiting time of one second is to improve the transient response of the parasitic power

on the data bus. At the end of the insertion of the measured values into the round robin database is done. The construction of the path to the database assumes that the script and the database are in the same directory.

The script is stored in the executable file gettemp.py and you may execute it on the command line. The result can be checked using rrdtool lastupdate. This ouputs the time stamp and the values of the last update of the database:

```
chmod +x gettemp.py
./gettemp.py
rrdtool lastupdate temperature.rrd
 temp0 temp1
1386777156: 18.937 5.687
```

Did hardware and software for a while diligently collected temperature readings, then RRDtool can create nice graphics, for example, visualize the history of the temperature of the last week as a line graph:

```
rrdtool graph tempweek.png \
  -s 'now - 1 week' -e 'now' \
  DEF:temp0=temperature.rrd:temp0:AVERAGE \
  LINE2:temp0#00FF00:Inside \
  DEF:temp1=temperature.rrd:temp1:AVERAGE \
  LINE2:temp1#0000FF:Outside
```

3. CONCLUSION

The proposed article implements a temperature measuring station based on the Raspberry Pi with minimal external hardware. The parasitic power supply of the sensors requires only a two-wire cable, but the maximum possible cable length is likely to be limited compared to an active power. If you need to measure temperatures above 70° C, then you should definitely provide an active power supply with a three-wire cable. The GPIO ports on the Raspberry Pi are connected without further buffering with the 1-wire bus. Therefore, the coupling of interference by parallel routed power lines is particularly to be avoided! The solution works for several month reliably with two sensors and a total cable length of about ten meters.

4. ACKNOLEGMENT

This paper was supported by project “Mathematical models and statistical methods for research on entrepreneurship and competitiveness of business in Southwest Bulgaria” funded by National Science Fund of Bulgaria

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A Microcomputer Data-Acquisition System for Real-Time Processing of Measured Data

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***Abstract**: A Microcomputer data-acquisition system for real-time processing of measured data is an instrument used to measurement of physical quantities. In this paper, a new data-acquisition and real-time processing system designed for measurements is presented. It is based on a portable microcomputer - named BeagleBone Black, which acquires and processes the data, records the size distribution every minute, and displays and prints the distributions and selected integrated parameters of the precipitation. The last part of the paper presents one real example for application of performed with the system on varied types of sensors.*

***Keywords:** Microcomputer, data-acquisition system, measurement of physical quantities, temperature;*

1. INTRODUCTION

BeagleBone Black is a low-cost, community-supported development platform for developers and hobbyists. Boot Linux in under 10 seconds and get started on development in less than 5 minutes with just a single USB cable. The Processor is AM335x 1GHz ARM® Cortex-A8 and 512MB DDR3 RAM, 4GB 8-bit eMMC on-board flash storage, 3D graphics accelerator, NEON floating-point accelerator, 2x PRU 32-bit microcontrollers. The Microcomputer BeagleBone Black was software compatibility with Debian, Android, Ubuntu, Cloud9 IDE plus much more operating systems. The connection with board is possible to be done via USB client for power & communications, USB host, Ethernet, HDMI and 2x 46 pin headers peripherals.[1]

2. METHODS

First step when we started with BeagleBone Black is update board with latest software. There are multiple ways to run initial software on your board, but it is likley that the simplest way to get an update is to create an exact replica of a bootable microSD card and boot off of it. The BeagleBone Black Rev C has 4GB of eMMC storage that can be initialized by a program booted

off of a microSD card. If you want to update to the latest software image for your board, this is a way to do that.

After that is time for BoneScript interactive guide. BoneScript is a JavaScript library to simplify learning how to perform physical computing tasks using your embedded Linux. The BoneScript library provides several functions useful for interacting with your hardware. Performing physical computing tasks in JavaScript is a rather different than C on microcontrollers. JavaScript and the Node.JS interpreter like to do everything asynchronously using callbacks. An event loop runs waiting on whatever the next system-blocking event is, such as waiting for a keypress or a file load to complete. The callbacks are then executed to completion before other event handlers are run. Timing operations in JavaScript are provided by setting timers with callback event handlers. A nice overview of JavaScript timers can be found on www.w3schools.com.

```
var timer = setTimeout(callback, milliseconds)
clearTimeout(timer)
var timer = setInterval(callback, milliseconds)
clearInterval(timer)
```

The BoneScript Library runs in Node.JS. You can run it directly on the board using the 'node' interpreter or the Cloud9 IDE (show on next page) that invokes the 'node' interpreter. You can also run it using the bonescript.js script within your browser via remote procedure calls using Socket.io and served up by the web server running on your Beagle.

Access to the library functions is provided through the "require('bonescript')" function call. The call returns an object containing all of the functions and constants exported by the library. The Node.JS API documentation on modules provides more information on the usage of 'require' within the 'node' interpreter.

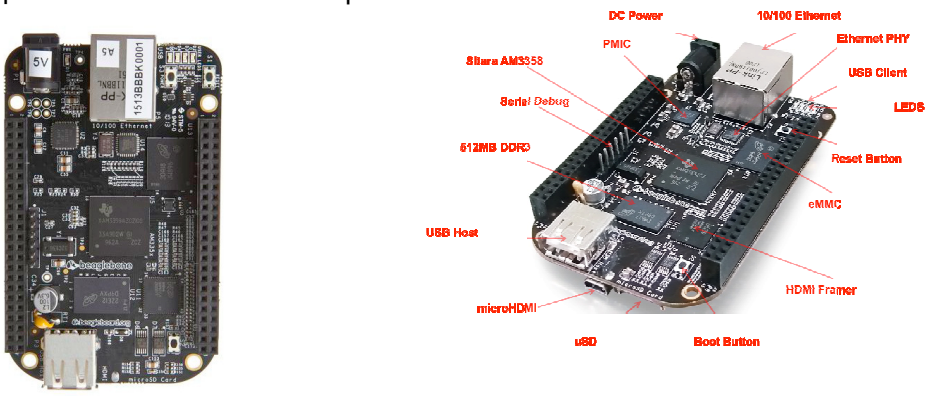


Fig.1. BeagleBone Black

To begin editing programs that live on your board, you can use the Cloud9 IDE. If your board is plugged into your USB port, click on the "Cloud9 IDE" link above to start the editor.

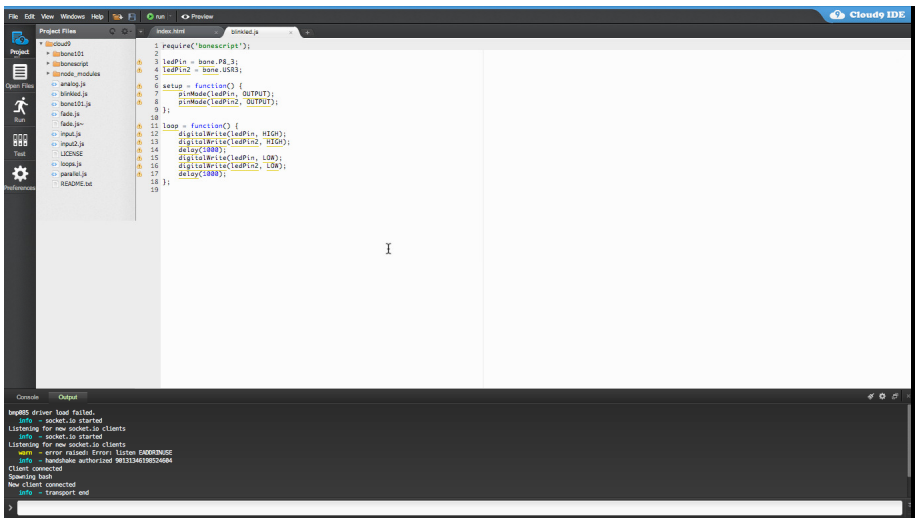
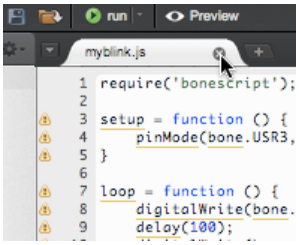


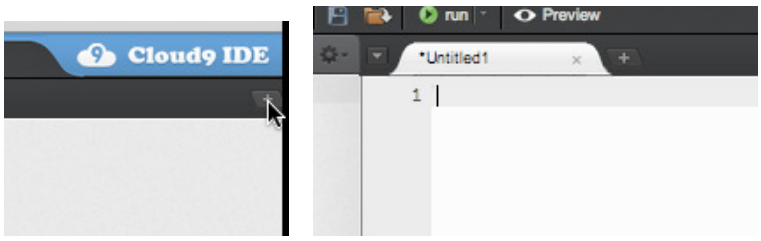
Fig.2. Cloud9 IDE

As a simple exercise to become familiar with Cloud9 IDE and the Bonescript JavaScript library, creating a simple application to blink one of the 4 user programmable LEDs on the BeagleBone is a good start.

- Step A: Close any open file tabs.



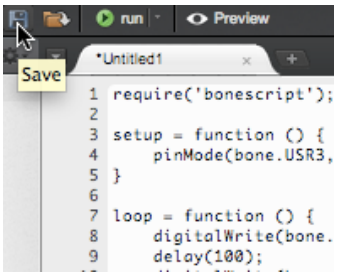
- Step B: Click the "+" in the top-right to create a new file.

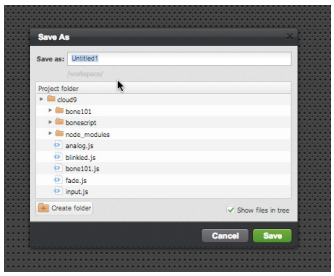


- Step C: Cut and paste the following code into the new tab:

```
var b = require('bonescript');

var state = b.LOW;

b.pinMode("USR0", b.OUTPUT);
b.pinMode("USR1", b.OUTPUT);
b.pinMode("USR2", b.OUTPUT);
b.pinMode("USR3", b.OUTPUT);
setInterval(toggle, 1000);

function toggle() {
  if(state == b.LOW) state = b.HIGH;
  else state = b.LOW;
  b.digitalWrite("USR3", state);
   }
```

- Step D: Save the file by clicking the disk icon and giving the file a name with the .js extension.





- Step E: Run the code by selecting the arrow to the right of "run" (or "debug") in the toolbar to pull down the list of files to run and select your new file.

- Step F: Observe the BeagleBone USR3 LED blinking steadily about 5 times a second.

- Step G: Stop the code by clicking "stop" in the toolbar.

Once you've finished developing your JavaScript application, you can have it start upon boot-up by simply dropping it into the 'autorun' subfolder (located at /var/lib/cloud9/autorun in the file system).
The systemd bonescript-autorun.service runs at start-up and uses the /usr/lib/node_modules/bonescript/autorun.js script to automatically detect when .js files are in this directory and invoke them as separate processes with

node.js. When the files are changed or moved, the script will kill the processes.

3. RESULTS AND DISCUSSIONS

Using a DS18B20 temperature sensor on a BeagleBone isn't difficult, but requires a bit of coding. Let's get started with an introduction on the different types of sensors. The actual DS18B20 sensor is around the size of a pea. You can order it pre-wired in a few different configurations, or purchase just the sensor itself. For my application, I am using the already waterproof sensor shown here. It is enclosed in a stainless thermowell and wrapped with shrinkwrap.

Fig.3. Standard DS18B20

You will need to connect a 4.7k resistor between the data wire and 3V wire. This diagram shows 5V, but we can still have it work using 3V. You only need one if you are using multiple sensors. More on this below.

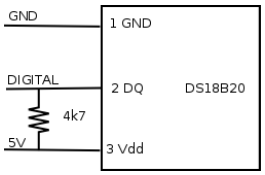


Fig.4. DS18B20 with resistor between the data wire and wire

Next, you will need to choose a GPIO pin to plug your DS18B20 into. I used this image to help visualize where each pinout was located . For this example, I used the P8 header, pin number 11.

Notice that the number of the pin does not correspond to the GPIO number! Also of importance, you cannot simply choose any pin. Some of the pins are reserved- for example, most of the higher numbered pins on the P9 header are taken up by the HDMI interface unless you have disabled it. Which pins are free are detailed in the next step. If you're familiar with raspian (the optimized debian distro running on Raspberry Pi's), you'll recall the setup is essentially plug and play thanks to raspian's kernel. On the Beaglebone, it's

consistent with the newer linux kernel so you will need to compile a device tree. At it's most basic level, a device tree simply informs the kernel about the location and type of a specific piece of hardware.

Now that we are wired up, boot your Beaglebone and login with SSH. We now need to compile a device tree for the one-wire interface. This is where noting the GPIO pin becomes important. Open the P8 or P9 PDF below and find the head pin you used. You may need to click 'Raw' if you are taken to Github. ow, let's get into some coding. Login via SSH. If you are using a different GPIO pin, you will need to make some changes. See below.

```
/dts-v1/;
/plugin/;

/ {
    compatible = "ti,beaglebone", "ti,beaglebone-black";
    part-number = "BB-W1";
    version = "00A0";

    exclusive-use = "P8.11";

    fragment@0 {
        target = <&am33xx_pinmux>;
        __overlay__ {
             bb_w1_pins: pinmux_bb_w1_pins {
                 pinctrl-single,pins =  <0x34 0x37 /* gpmc_ad13.gpio1_13, OMAP_PIN_INPUT_PULLUP | OMAP_MUX_MO
             };
        };
    };

    fragment@1 {
        target = <&ocp>;
        __overlay__ {
            onewire@0 {
                status          = "okay";
                compatible      = "w1-gpio";
                pinctrl-names   = "default";
                pinctrl-0       = <&bb_w1_pins>

                gpios = <&gpio2 13 0>;
            };
        };
    };
```

Fig.5. Coding for data-acquisition system

If you are using a pin other than P8_11, you will need to make a few changes. See the PDF header tables linked above for GPIO information.On line 19, change exclusive-use = "P8.11"; to the corresponding header and header pin you used. On line 25, change 0x34 to your corresponding GPIO offset. The offsets are found in the PDF's above. For example, if I used P8_12, the code should be 0x30. Leave 0x37 where it is. Save your edit and return to the shell.

Now we can build the file using dtc:

```
dtc -O dtb -o w1-00A0.dtbo -b 0 -@ w1.dts
wget -c https://raw.githubusercontent.com/
RobertCNelson /tools/master/ pkgs/dtc.sh
```

Copy it to /lib/firmware and load:

```
cp w1-00A0.dtbo /lib/firmware
echo w1 > /sys/devices/bone_capemgr.9/slots
```

You should be set now. To ensure everything has loaded correctly, run the following command. If everything is correctly configured, you should see Override Board Name in your list.

```
$ cat /sys/devices/bone_capemgr.9/slots

 0: 54:PF---
 1: 55:PF---
 2: 56:PF---
 3: 57:PF---
 4: ff:P-O-L Bone-LT-eMMC-2G,00A0,Texas
Instrument,BB-BONE-EMMC-2G
 5: ff:P-O-L Bone-Black-HDMI,00A0,Texas
Instrument,BB-BONELT-HDMI
 7: ff:P-O-L Override Board Name,00A0,Override
Manuf,w1
```

You will need to run the echo w1 > /sys/devices/bone_ capemgr.9/slots command on every boot. To do this automatically, add the line to your /etc/rc.local file somewhere after the comments section.

Your temperature sensors are located at /sys/devices/w1_bus_master1/. They are of the format 28-00000xxxxxxx and correspond to your sensor's unique serial number.

To view the current value just cat the file. You will have two lines of output. The temperature (in celsius) is at the very end.

```
$cat/sys/devices/w1_bus_master1/28-0000045d9d8a/
w1_slave
07 01 4b 46 7f ff 09 10 da : crc=da YES
07 01 4b 46 7f ff 09 10 da t=16437
```

The current reading is 16.4 degrees celsius.

4. ACKNOLEGMENT

This paper was supported by project "Mathematical models and statistical methods for research on entrepreneurship and competitiveness of business in Southwest Bulgaria" funded by National Science Fund of Bulgaria

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The Benefits of Applying Formative Assessment Techniques for Better Student Learning

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Abstract: *The subject of the following study is preparation of a comparative analysis of the results of a group of students that have been thought based on two different educational standards - traditional techniques for teaching and assessing in Bulgaria and an alternative method which includes techniques described in the book of Dylan William "Embedded Formative Assessment". This comparative characterization and analysis could be used as a foundation for drawing some basic conclusions for the general state and success rate of the reviewed educational methods and more particularly to be used as a review of the extend of competitiveness of the Bulgarian educational system. In some cases, based on the study, some general trends could be commented upon as well as some conclusions could be drawn and even some foundations for future implementation of successful educational strategies could be established.*

Key words*: Formative Assessment, Summative Assessment, Assessment for learning*

1. INTRODUCTION: NATURE OF FORMATIVE ASSESSMENT

One of the most widely-recognized and discussed characteristics of assessment nowadays has been the change from the traditional teaching and assessment techniques and strategies which are proved to provide only a limited influence on the ongoing learning experiences of students and in the direction of putting the emphasis on the relationship between assessment and the process of classroom learning.

Therefore, it has been keenly hypothesized by educational methodologists and experts that by increasing the effectiveness of classroom assessment, student learning will markedly improve as well. This could be achieved by the ongoing interaction of both students and teachers in the action of tracking the learning achievements of the first group and adjusting the teaching methods used by the second group to more successfully

implement the educational process. This concept has been known as "formative assessment" or "assessment for learning" and has been recognized by many teachers worldwide as the most efficient instrument for filling in the gaps in students' understanding in a particular subject area (Math/Science/English literature etc.) and respectively improving their results from learning.

There is no strict definition of a "formative assessment" and a broad literature review reveals several interpretations by experts in the area. According to Black and Wiliam, the forefathers of the initial study on the subject, it is not only restricted to a tool or a single event, but rather a collection of practices with a common characteristic:

"All those activities undertaken by teachers, and by their students in assessing themselves, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged. Such assessment becomes 'formative assessment' when the evidence is actually used to adapt the teaching work to meet the needs." (Black & Wiliam, 1998)

"Formative assessment, therefore, is essentially feedback (Ramaprasad,1983) both to the teachers and to the pupil about present understanding and skill development in order to determine the way forward" (Harlen & James, 1997, p. 369).

"Formative assessment refers to assessment that is specifically intended to provide feedback on performance to improve and accelerate learning" (Sadler, 1998, p. 77).

"An assessment is formative to the extent that information from the assessment is feedback within the system and actually used to improve the performance of the system in some way" (Wiliam & Leahy, 2007, p. 31).

"Formative assessment is defined as assessment carried out during the instructional process for the purpose of improving teaching or learning. What makes formative assessment formative is that it is immediately used to make adjustments so as to form new learning" (Shepard, 2008,p. 281).

2. COMPARISON BETWEEN FORMATIVE AND SUMMATIVE ASSESSMENT

Formative assessment provides a detailed diagnostics of the extend to which the ongoing instructional process is successful, while learning is taking place, and while learning is occurring. Although formative assessment gives information and data during the instructional process regarding the student's individual learning accomplishments while the teaching takes place in class,

it also can provide information on the teacher's own progress at delivering the lesson, therefore, proper modifications could be carried out. Its fundamental goal is to identify areas in need of improvement by creating a safe atmosphere in class where student learning is constantly tested through various techniques in search of mistakes and gaps which then could be clarified by the instructor.

Types of Formative Assessment

- Observations and effective feedback during in-class activities
- Giving a pretest
- Reflections journals that are reviewed periodically during the semester
- Question and answer sessions, both formal—planned and informal—spontaneous
- Conferences between the instructor and student at various points in the semester
- In-class activities where students informally present their results
- Student feedback collected by periodically answering specific question about the instruction and their self-evaluation of performance and progress.

Every teacher is familiar with the nature and purpose of the summative assessment. It is used as a tool to measure student mastery and understanding of a particular unit after it has been taught and discussed in class. It gives an accurate measure of the level at which a student has advanced and it also identifies areas of improvement and further work. The final result of the summative assessment is a grade which reflects the level of learning that has taken place after the student has been provided with sufficient amount of instructional time, practice and opportunities for question-answer sessions and in-class discussions. It is a snap-shot of a student success or failure to master a certain topic or concept but it fails to reflect their overall learning experience and most importantly how it could be adjusted to better reflect student learning needs.

Types of summative assessment might include, but are not limited to, the following:

- a final oral/visual/multi-media presentation,
- a final draft of essays/paper/drawing (etc.),
- a final project/performance/demonstration/exam,
- a mid-unit quiz/test/performance,
- an end-of-unit test/exam/performance
- a locally or state developed standardized assessment

It has to be made clear that the two types of assessment are not mutually exclusive and contradictory to one another. In fact best teaching practice suggests that formative assessment needs to be incorporated in the lesson plans on daily basis because it gives the opportunity to gather data on student performance, difficulties and failures of understanding and make proper adjustments prior to the final summative assessment thus increasing student success rate. This comparison between the two types of assessments could be illustrated by the following diagram:

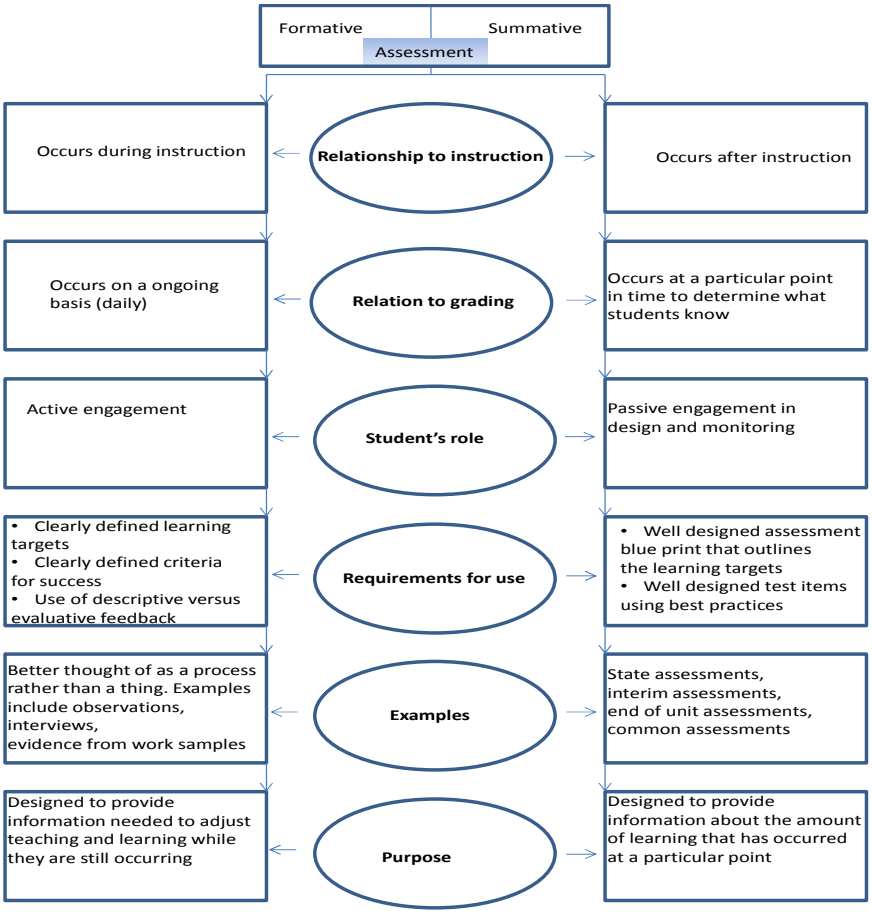

Fig 1: Comparison between the two types of assessments

3. CASE STUDY

The experiment was carried out at the American College of Sofia during 2013/2014 academic year. The grading period is divided in two semesters each of which is separated in quarters, four altogether. The semester grade included 40 percent of each quarter grades and 20 percent of a semester one final exam in the subject area. The year grade is the average grade from the two semesters.

Formative assessment has been an active part of the professional development program at the school. Based on the positive feedback from literature, formative assessment was applied during the last three quarters for the entire 11^{th} and 12^{th} grade levels in 2013/2014 academic year and then the results were compared to those of the 11^{th} and 12^{th} grade levels students in 2012/2013.

The following number of formative assessment techniques was used:

Technique Description	Formative benefit from the technique
Exit/Entrance tickets Based on the teacher's extensive experience on the material and the most frequent mistakes made by students, in the beginning of each lesson students were given a piece of paper with a math problem. After an instructional part presented by the teacher and sufficient number of problems being solved in class, by the end of the period each student was given five minutes to solve the problem individually. Upon leaving the classroom each student had submitted their "exit tickets" to the teacher. Students who had tried to submit a blank piece of paper were asked to go back to their desks and work further on the task.	The "exit tickets" are checked by the teacher and upon students entering the classroom next time a seating chart is presented in the room. Students sit in groups of four, having approximately two students who have successfully solved the problem and two students who have made mistakes. All students provide arguments within the group regarding their solution. This activity lasts approximately five minutes at the end of which a final summery/discussion of the correct answer is provided by the teacher.

Traffic lights Each student is given three cups – green, yellow and red. In the beginning of the lesson all students have their green cup on top of the others on their desks. This indicated that they still have no problems with the material that is being presented. Students are instructed to place the yellow cup on top of the others to alarm the teacher that they might be going too fast with delivering of the new topic and they need to repeat their last point. Students place the red card on top of the others only in cases when they are completely stuck in a particular problem and could not continue working without individual attention from the teacher.	The "traffic lights" technique has proven to be extremely useful since the teacher immediately can recognize if they need to slow down the pace of the lesson or in some instances once again repeat their point or support it with further examples. Students who have used too frequently their red cups are invited on individual weekly consultations.
Laundry day This technique is used before a unit test. The desks in the classroom are rearranged in work stations each of which deals with a particular type of math problem. There are usually four to five separate work stations. In the beginning of the lesson each student sits at the workstation which deals with a type of problem that he finds most problematic. After solving sufficient numbers of problems together in a group, the student is free to move to another workstation. In the last ten minutes each student is handed with a very simple sample of the four/five types of problems which they need to solve and submit to the teacher.	Students receive feedback from the teacher regarding their mistakes and a certain number of similar math problems which they can solve at home for further practice.

4. RESULTS FROM THE EXPERIMENT AND CONCLUSION

Results from the previous 2012/2013 year (shown in Tab1 below) demonstrate that students experienced certain constrains with the last two topics from the curriculum which involve Logarithmic equations and inequalities and Trigonometry (both functions and trigonometric equations and inequalities). The cause of those observations could be complex but it was hypothesized that the decrease of performance was a result from the

upcoming AP exams 11th graders took during that time and the decrease of motivation due to the end of the school year. To eliminate those factors it was decided to reshape the curriculum for the current 2013/2014 academic year and to cover those topics earlier and leave the topic of Statistics, which students comprehend more easily due to it practical application, for the end of the year, while in the curriculum of the Seniors no changes were made. On the next table and diagram the comparison between 2012/2013 and 2013/2014 academic year's results could be seen.

Academic Year 2012/2013 (sample group 1)					Academic Year 2013/2014 (sample group 2)				
Class	Number of Students	Average Grade on Entry level test	Average Grade on Final level test	Progress	Class	Number of Students	Average Grade on Entry level test	Average Grade on Final level test	Progress
11/1	20	5,50	5,50	0,00	11/1	20	5,12	5,24	0,12
11/2	20	5,26	5,15	-0,11	11/2	20	5,31	5,51	0,20
11/4	20	5,66	5,40	-0,26	11/4	20	5,29	5,36	0,07
11/5	20	5,40	5,23	-0,17	11/5	20	5,23	5,26	0,03
12/1	18	5,40	5,24	-0,16	12/1	18	5,30	5,44	0,14
12/2	18	5,31	5,20	-0,11	12/2	18	5,12	5,30	0,18

Tab1: Comparison between 2012/2013 and 2013/2014 academic year's progress

From the Tab1 it is visible that with classes to which formative assessment has been applied (sample group 2) there is a progress which varies between 0.03 to 0.20 increase of their average grades while in sample group 1 there is either no progress or even a decrease (between 0.11 to 0.26) in the results of their Final exam. Although formative assessment does not compensate for all gaps in the educational process, the results from our study suggest that by using it consistently on weekly basis students feel more confident and more willingly engage in classroom activities as well as show higher academic achievements. It further enhances students' ability to create interdisciplinary connections, work and support each other in group projects and activities. The benefit from the regular use of formative assessment is also highlighted by the higher results of the teacher on their annual feedback survey. Students comment in their feedback forms that they find the lesson plans more engaging, attractive and stimulating because of the use of formative assessment. Students report in their feedback form that the application of formative assessment has helped them develop a host of skills that are increasingly important in the professional world - break complex

tasks into parts and steps, plan and manage their time more effectively, refine understanding through discussion and explanation and develop stronger communication skills.

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MINI-CONFERENCE

"Inquiry-Based Approach in Higher Education in Mathematics and Informatics"

Serious Computer Games Design as Inquiry Based Learning in Teacher Education

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(Plenary report)

***Abstract**: Inquiry-based learning is a pedagogical method, based on constructivist learning theory where learners carry out research, set questions, make observations, collect, analyse, and interpret data, outline explanations, and create predictions. Serious games design can provide a framework to support confirmation, structured, and guided inquiry. There is a convergence between the core elements of a good serious game design and the characteristics of productive learning. Another link between games and learning is formative feedback as a critical part of any learning effort and a key component in game design that adjusts challenges.*

***Keywords:** serious games design, ICT, inquiry-based learning, learning goals, teacher education, formative feedback*

1. INTRODUCTION

Learning in schools is still heavily geared toward the acquisition of content within a teacher-centered model, with instruction too often abstract and decontextualized and thus not suitable for this age of complexity and interconnectedness. This is especially topical for "*digital generations"*, born after 1985. They grew up surrounded by digital media, and most of their activities dealing with peer-to-peer communication and knowledge management, in the widest sense, are mediated by these technologies. The changing ways of how members of this generation learn, communicate, and entertain themselves are the primary reasons behind the growing popularity of social computing applications, sociability, and team spirit, engagement and working attitude, multitasking, individualization and personalization, immediacy, and fluency with multiple media types [10], [12]. But these generations are also called "*gamer generations"* as they spend a lot of their time playing computer games. And as there are many connections between games and learning, educators could integrate games into teaching and learning. Game design has a lot to teach us about learning, and contemporary learning theory has something to teach us about designing better games [17]. Marshall McLuhan, famous Canadian philosopher of communication theory, who predicted World wide Web in sixties, when he talked about a "global

village", stated: "*Anyone who makes a distinction between games and learning doesn't know the first thing about either*."

Games can lead to changes in attitudes, behavior, and skills and that is actually exactly that how *learning* is defined. We need to explain why game based learning is engaging and effective and when, with whom and under which conditions games can be integrated into learning.

There is a convergence between the core elements of a good game and the characteristics of productive learning [16]. The constructivist problem-based and inquiry-based learning methods indicate the success of learning in the context of challenging, open-ended problems. Goal-based scenarios have long been viewed as an active primer for situated learning. Correspondingly, in a good game a player is involved in an iterative cycle of goal-based, interactive problem solving. Another link between games and learning is formative feedback - a critical part of any learning effort [15], and also a key component in good game design that adjusts challenges and gives feedback so that different players feel the game is challenging and their effort is paying off. Well-designed games have the potential to support meaningful learning across a variety of content areas and domains. But all games are not good for all learners and for all edu goals!

Gee claimed [4] that the secret of a good game was not its 3D graphics with bright colors and exciting sounds or music, but its underlying architecture where each level is adapted to the outer limits of the player‘s abilities, seeking at every point to be hard enough to be just doable. Cognitive psychologists have long claimed that the best instruction hovers at the boundary of a student‘s competence. And why aren‘t the games used more widely in classrooms if they are so useful for learning?

While time constraints, cost and availability of appropriate games, and a lack of knowledge about their integration into learning are possible reasons for this, the major problem is the lack of good research on games and learning [18]. Compared to other types of instructional systems, there are too few experimental studies examining the range of effects of gaming environments on learning, and a corresponding lack of theory and practice for their design and implementation.

2. GAME BASED LEARNING

There are several reasons that draw educator attention to games [5]. In formal education we experience a shift from traditional didactic model, which is focused on instruction, to learner-centered model that emphasizes the active learner's role. We also changed the view of learning goals from lower taxonomic levels, like just recalling information, to higher levels, such as finding and using of information in a new settings.

Prensky [11], Gee [4], and Whitton [19] defined game based learning as *a process of learning with the use of digital games*. Games can provide motivation for learning, thus increasing the chance that the desired learning outcomes will be achieved. Learning is defined as the acquisition of knowledge or skills through experience or practice, and what better way to learn than through a game [9]. Almost all studies about game based learning show that students are highly motivated when learning materials are presented in a computer game format and that this has positive effects on the acceleration of a learning process [20], [21]. Students need motivation to focus on what needs to be learned but for any quality learning to occur this is not enough. Comparing learning outcomes of students who learned with computer games and those learning with another type of learning materials shows that there is no significant difference between them. This is usually because of inappropriate game design. Games can be very appealing to students but if they entertain and not teach, the use of games in education does not make much sense. So we have to find out what are the elements that make computer game an educational computer game.

Gross [6] claims that digital games for educational purposes must have well defined learning goals and have to promote development of important strategies and skills to increase cognitive and intellectual abilities of learners.

According to Malone [8] and Garris [3], the elements contributing to educational values of digital games are sensual stimuli (visual and audio representations of learning material), fantasy (context presented in imaginary setting), challenge (demanding or stimulating situation) and curiosity (desire to know or learn). These elements must be incorporated on an integrated platform, to structure objectives and rules, a context of meaningful learning, an appealing story, immediate feedback, a high level of interactivity, challenge and competition, random elements of surprise, and rich environments for learning.

A game can be instantiated for learning as it involves mental (and sometimes physical) stimulation and develops practical skills – it forces the player to decide, to choose, to define priorities, to solve problems, etc. Immediate reward (and feedback) is a major motivational factor, whether it is translated as game entities (more life power, access to new levels, etc.) or as neurological impulses (happiness, feeling of achievement, etc.). Games can be social environments, sometimes involving large distributed communities. They imply [1] self-learning abilities (players are often required to seek out information to master the game itself), allow transfer of learning from other realities, and are inherently experiential with the engagement of multiple senses .

Van Eck [18] argues that games and play can be effective learning environments not because they are fun but because they are immersive,

require player to make frequent important decisions, have clever goals, adapt to each player individually and involve social network.

If we consider a model of game based learning by Garris [3], the main characteristic of educational game is that instructional content is blurred within game characteristics. Students are playing the game and having fun, forgetting about the "learning" part of the experience even though they are constantly presented with new concepts which they have to adapt in order to be successful in game. We should foster motivation with game design promoting repeating the cycles within game context. Player is expected to elicit desirable behaviors based on emotional and cognitive reactions that result from interaction with and feedback from gameplay.

Gee [4] argues that features of video games with high learning potential fall into two categories: 'non-game' features, which may also appear in non-game contexts, and 'game' features, which relate more to the 'gameness' of games. Despite this distinction, it should not be assumed that the 'non-game' features would work as well for learning if they were detached from the 'game'.

'Non-game' features of games with high learning potential are:

- Empathy for complex systems – to look at complex system from the inside in order to understand how its variables are interacting.
- Simulations of experience and preparations for action – in video games, players see the virtual world in terms of how it affords the sorts of actions they need to take to accomplish the goal of winning.
- Distributed intelligence via the creation of smart tool - good video games distributes intelligence between a real-world person and artificially intelligent virtual characters in order to represent macro and micro view of the situation.
- Cross-functional teamwork – Good video games may be able to teach collaboration and cross-functional teamwork for institutions like schools and workplaces. In multiplayer games like World of WarCraft groups are composed of different character types, such as hunter, warrior or priest, who each play the game in a different way. Players interact with each other not in terms of their real-world characteristics but through their functional gaming identities. They may also choose to use their real-world race, class, culture, and gender as strategic resources but they are not forced into pre-set racial, gender, cultural or class categories.
- Situated meaning – Dialogue and experience are essential for people to be able to relate words to actual actions, functions, and problem solving. Since video games are simulations of experience, they can situate language in specific contexts.
- Open-endedness: melding the personal and the social – In good open-ended games, players construct their own goals, which are based on

their own desires, styles and backgrounds. Combination of personal and in game goals produces a state of high motivation.

Features of a 'good game' that relates to effective learning are the following:

- Motivation
  Video games are profoundly motivating for players, and it is important to understand the sources of this motivation if it is to provide a foundation for learning.
- The role of failure
  The price of failure is lowered and is often seen as a way to learn the underlying pattern. These features of failure in games allow players to take risks that might be too costly.
- Competition and collaboration
  Many gamers, including young ones, enjoy competition with other players in games but may not see competition as pleasurable and motivating in school. Competition in video games is seen by gamers as social, as much about gaming as winning and losing.
- The design of games that relates to:
  - Interactivity – player doesn't just passively consume knowledge but has control over content;
  - Customization – based on learning styles and providing multiple routes to success;
  - Strong identities - Good games offer players identities that trigger a deep investment on the part of the player and which are clearly associated with the functions, skills and goals one has to carry out in the virtual world;
  - Well-sequenced problems - In connectionist approaches to learning, it is argued that sequencing is crucial for effective learning in complex domains;
  - A pleasant level of frustration - adjusting challenges in such a way that a range of players can experience the game as challenging but do-able;
  - A cycle of expertise - repeated cycles of extended practice and tests of mastery;
  - 'Deep' and 'fair' – game must be challenging but set up in a way that leads to success.

Gameplay elements should be initially simple and easy to learn and become more complex the more the player comes to master them.

3. SERIOUS GAMES DESIGN IN TEACHER EDUCATION

Rieber, Smith and Noah [13] stated already in 1989 that there are two distinct applications of games in education: game playing and game designing. Game playing is the traditional approach where one provides ready-made games to students. Game designing assumes that the act of building a game is itself a path to learning, regardless of whether or not the game turns out to be interesting to other people. The idea of "learning by designing" is based on the assumption that active participation in the design and development process is the best way to learn something. This approach has gained increased prominence due to the proliferation of computer-based design and authoring tools.

At the Faculty of Education, University of Ljubljana we started to implement this approach six years ago in the framework of the two-semester course *Application of ICT in education* for future computer science teachers. There have been a lot of different approaches, methods and ideas developed on how to organize the process of designing and developing educational game from the initial idea into to final product. We could not agree completely with any of them so we decided to develop our own method, called 'SADDIE' [14].

3.1. SADDIE methodology for serious game design

Our main focus was on creating educational games with a high learning value but it turned out that the method had in fact two important outcomes. The first outcome is serious game itself. This is only a side effect of the main goal, which is to motivate our students to work actively and to learn in an efficient way through carefully refined process of active engagement in the game design and production process. The second outcome is students' improvement of the competences that are crucial for teachers. Such competences include the ability to determine learning objectives that are consistent with the curriculum, the selection of appropriate teaching approaches and their implementation in learning process, preparation of feedback, evaluation of acquired knowledge and evaluation of the learning process.

Serious games design is implemented as a project in a course with a total of 8 ECTS credit points. It means that the students are supposed to spend between 220 to 240 hours in the project activities. The main idea of the course is to combine all didactic and technical knowledge that students acquired during their studies at the faculty and to apply it in a relatively complex project. According to constructivist learning theory, the course has very limited number of traditional lectures. Only the main project requirements are presented by the lecturer in the introductory phase of the project and some general rules about the learning goals and about organization of work are

defined. Students then have to follow 'SADDIE' methodology that defines framework phases and other incidental activities.

Students work in groups of 3 or 4 students. Students are supposed to organize their activities by themselves. They are free to define different types of organizations of work and to accept different roles. Later, during the execution of the project, students in project groups write a log in which they report on the dynamics and organization of the work of the group. By analyzing the logs, we found that in some groups there were exposed leaders and in the others the responsibility was evenly distributed among all members.

At the regular weekly meetings groups prepare oral reports on the work in the past week and on any problems they have encountered. They receive immediate feedback from peers and from the teacher. If difficulties arise, the teacher initiates discussion about what could be the reasons for a problem and gives some hints or suggests possible ways to solve them.

3.2. Main project phases in SADDIE

SADDIE is an acronym of six main project phases: Specification, Analysis, Design, Development, Implementation, and Evaluation. SADDIE is an extension of 'ADDIE' instructional design approach that was developed at the Florida State University in the seventies and was used for designing learning materials for American army [2].

3.2.1. Specification

The game design begins with the *specification phase* where students choose the topic from computer science curriculum for primary or secondary school. They are instructed to identify learning goals that are complex and challenging for students or the ones that are very important in general. They have to consider if the students could benefit from presenting the topic in a game format and make a didactic foundation for later stages of the project. It is highly recommended that students roughly define the game basics in this phase: time and place of the events, characters, artefacts, challenges, rules, and goals. They should also ensure that game is funny, it provides some sort of competition or conflict, and it is entertaining and recreational. After this initial phase they have to submit a document in which they present: a short presentation of the selected topic with arguments for their decision, a brief description of a gameplay, a list of specific learning goals from the official curriculum, motivational elements, specific methods for achieving the learning objectives (i.e. method of: explanation, discussion, practical application, clarification…), the assessment of learning, tools for communication, collaborative work and the suggestions on how to incorporate game into learning process.

Some examples of the selected topics in the previous years are: 2D arrays in Pascal Programming language, Dijkstra algorithm, the meaning of functions and procedures in programming, comparing efficiency of different sorting algorithms, responsible behaviour when using social networks, software threats on internet, …

3.2.2. Analysis

The analysis phase focuses on analysing the specification and preparing all the information needed for implementation phase. First decision is to select the target audience for the game that directs the choice of game story, level of difficulty, graphics, characters, interface, type of interactions and gameplay itself. Students have to consider available resources for implementing their ideas and make compromises. This includes selecting appropriate game engine, software for creating visual and audio artefacts and animations. Students have to focus on defining expected taxonomic levels of knowledge for selected learning goals. This is followed by the most important stage of analysis phase: mapping learning goals with game goals. The main idea is to blend learning content into game context so that no important details are lost or that translation would confuse or blur the learning content. The goals of a game should support understanding of the main concept in a different, more exciting way that is familiar to target audience and as such can ease the knowledge transfer. Progression throughout the game is possible only if player properly understand the concept behind game goal.

3.2.3. Design

The goals of a game can be associated with different types of knowledge. We are using the Kapp's taxonomy [7] that defines the following categories: declarative, conceptual, and procedural knowledge, knowledge based on rules and some other, so called ‘soft skills’. Each goal of a game is evaluated based on this classification and incorporated into the story. detailed scenarios are prepared In the design phase, with dialogues that integrate the requirements and recommendations from the specification phase. Students are then required to design all the graphical elements needed for the games, such as backgrounds for the scenes, artefacts and all characters. They are also encouraged to record speech, sounds and music for the game and to make simple animations.

3.2.4. Development

In the development phase, a game is produced with the selected game engine. When we started with educational game projects five years ago, students had to implement their own game engines with Actionscript 3 and produce the final product. We noticed that they spend a lot of time and energy on programming,so they couldn’t focus so much on the content and didactical

challenges. Our idea was to find the most appropriate game engine that would ease and fasten the development process. We decided to choose the e-Adventure platform, which has been developed in a research project aiming to facilitate the production of educational games at the Universidad Compultense de Madrid. It provides efficient tools for developing complex adventure games with just basic understanding of programming.

3.2.5. Implementation

Implementation phase defines various possibilities on how to incorporate learning with educational game into a learning process. Students have to consider different options and write specific proposals for teachers. It is crucial to find reasonable situations where learning outcomes can hardly be aachieved using traditional teaching methods. Usually different accompanying activities need to be prepared. Students carry out all these activities during their practical training in schools.

3.2.6. Evaluation and testing

The evaluation phase is taking place in parallel with the implementation or immediately after this phase. Each project group prepares beta testing that is carried out by their peers. This is followed by gamma testing during their practice in primary or secondary schools. Students measure the efficiency of the alternative learning approach, comparing it to the traditional one, and they get feedback from the players and from teachers. This allows them to find out whether the objectives from Specifications have been achieved and to improve the quality of their products.

3.3. Achieving teaching competences

SADDIE method is designed to foster the process of improving teaching competences that are required for our students to be able to independently design any instructional material [14]. Students achieve *didactic and technical competences* as well as *skills for working in a team*. Didactic competences that student are expected to achieve during the above described process are: determining learning objectives that are consistent with curriculum, reflective consideration on selection of learning goals, classification of learning goals according to taxonomic levels and type of knowledge, selecting appropriate activities that support learning on preferred taxonomic level, identifying the concept behind learning objective and ability to map the core idea into another context without losing any important information and preparation of didactically sound feedback. Students also learn how to properly evaluate their work, acquired knowledge, and learning process. One of the most important considerations is about incorporating educational game into learning process. The course is not focused on technical aspects of designing a game, but students usually become so motivated during the process that

they individually learn about animating, manipulating visual materials, recording and editing music and sound, drawing in perspective and learning more advanced programming techniques in order to make their projects more interesting. The ability to work in a team is a relevant competence in today's society. It is particularly important for the teaching profession, so we paid special attention to this aspect. Groups had to report on different aspects of collaboration in regular weekly meetings and all important activities and reflexions are collected in the log, which is written by each group during the project.

4. CONCLUSIONS

An inquiry based learning approach based on serious game design has been developed at the Faculty of Education, University of Ljubljana. Teacher students in the fourth year of undergraduate study program *Two subjects teacher* design in develop educational games in small groups in projects that are carried out in the framework of the two-semester course *Application of ICT in education*. The methodology for this approach has been developed and its application in the last six years had various positive outcomes. Serious games as final products are just a side effect of the more important goal, i.e. achieving different competences that are crucial for teachers. Such competences include the ability to determine learning objectives that are consistent with the curriculum, the selection of appropriate teaching approaches and their implementation in the learning process, development of a game, preparation of feedback for students, evaluation of acquired knowledge, evaluation of the learning process, independent planning and organization of all necessary activities in a project, and the ability to work in a team.

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Some Examples of the Research Approach in the Education of Mathematical Analysis

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(Plenary report)

I like crossing the imaginary boundaries people set up between different fields — it's very refreshing,

Maryam Mirzakhani, 2014 Fields Medalist

***Abstract**: We present three examples of a research approach in the education of Mathematical analysis for undergraduate students. The first example is for the inverse trigonometric and hyperbolic functions and the arithmetic operations. Several identities are presented and derived using Maple. The second example is the brachistochronne problem, cycloids and the origins of Calculus of variations. The third example is on fractional differential equations and numerical experiments with Mathematica.*

***Key words:** Inverse functions, Brachistrochrone problem, Calculus of variations, Fractional differential equations, Existence results, mathematical experiments*

1. INTRODUCTION

The goal of this paper is to show how the modern mathematics software systems can be applied to treat curious mathematics problems and attract students to research studies. Making "mathematical experiments" without necessarily deep knowledge can open the way to understanding nontrivial phenomena and initiate investigations. We present three problems of our educational and research experience. They show how the "mathematics experiments" can help for treating and giving examples to considered problems. The problem, how for a given natural number N one can express the numbers N+1, N(N+1), N-1 and (N-1)/N without using the arithmetic operations is considered in Section 2. It can be made using trigonometric and hyperbolic functions and their inverse functions. The Section 3 is devoted to the brachistrohrone problem and the beginning of Calculus of variations. The solution of the problem is a cycloid joining two given points. It is discussed

how the cycloid can be plotted. In Section 4, a fractional differential equation with impulses is considered. An example is given which is created by a "mathematical experiment". Historical notes on the beginning of Fractional calculus are also given.

2. INVERSE HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS AND ARITHMETIC OPERATIONS

Let N be a natural or positive number. In this section we discuss the presentations of the numbers N+1, N(N+1), N-1 and (N-1)/N without using the arithmetic operations: Plus (+), Minus (-), Times (x), Divide (/). It can be made using trigonometric and hyperbolic functions and their inverse functions. We suppose that the reader and the student is familiar with the trigonometric functions and their inverses.

Proposition 1. *We have for* $N \geq 0$

(1) $$N+1 = \left(tg\left(arc\,cot\,g\left(sin\left(arc\,cot\,g\sqrt{N}\right)\right)\right)\right)^2,$$

(2) $$N+1 = \left(tg\left(arc\,cot\,g\left(cos\left(arctg\sqrt{N}\right)\right)\right)\right)^2.$$

Proof. It follows by the definitions that

$$arc\,cot\,g(z) = arctg\left(\frac{1}{z}\right), \quad z > 0..$$

Then, for $z = sin(y) \in (0,1)$

(3) $$arc\,cot\,g(\sin y) = arctg\left(\frac{1}{\sin y}\right).$$

By a substitution $y = arc\,cot\,g\left(\sqrt{N}\right) \in (0,\pi/2]$, $N \geq 0$, (3) and

$$\sqrt{N+1} = \sqrt{cot\,g^2 y + 1} = \frac{1}{\sin y}$$

we obtain

$$arctg\left(\sqrt{N+1}\right) = arc\,cot\,g\left(sin\left(arc\,cot\,g\left(\sqrt{N}\right)\right)\right),$$

which implies (1). By analogy one can prove (2). ■

Using Maple one can derive following formulae:

$$tg\left(arc\,cot\,g\left(cos\left(arc\,cot\,g\sqrt{N}\right)\right)\right)=\sqrt{\frac{N+1}{N}},$$
$$tg\left(arc\,cot\,g\left(cos\left(arctg\sqrt{N}\right)\right)\right)=\sqrt{N+1},$$
$$tg\left(arc\,cot\,g\left(sin\left(arctg\sqrt{N}\right)\right)\right)=\sqrt{\frac{N+1}{N}},$$
$$cot\,g\left(arctg\left(sin\left(arctg\sqrt{N}\right)\right)\right)=\sqrt{\frac{N+1}{N}},$$
$$cot\,g\left(arctg\left(cos\left(arc\,cot\,g\sqrt{N}\right)\right)\right)=\sqrt{\frac{N+1}{N}},$$
$$cot\,g\left(arctg\left(sin\left(arc\,cot\,g\sqrt{N}\right)\right)\right)=\sqrt{N+1}.$$

To express N(N+1) one can use $N^2+1=(tg(arc\,cot\,g(sin(arcctgN))))^2$, and then apply N-1 times (1) or (2).
To express N-1 and (N-1)/N by N, we will use the inverse hyperbolic functions. Recall their definitions and properties.

1. The hyperbolic sine is the function $y=shx=\frac{e^x-e^{-x}}{2}$ **with domain R and range R.** For $z=e^x>0$ we have $z^2-2yz-1=0$, which implies $z=e^x=y+\sqrt{y^2+1}$ and $x=ln\left(y+\sqrt{y^2+1}\right)$ is the inverse function. **The inverse hyperbolic sine function** $y=arcshx=ln\left(x+\sqrt{x^2+1}\right)$ **has domain R and range R.**

2.The hyperbolic cosine function is $y=chx=\frac{e^x+e^{-x}}{2}$ **with domain R and range [1,+∞).** It is an even function and not invertible, but its restriction on [0,+∞) is increasing and invertible. For $z=e^x\geq 1,\quad x\geq 0$ we have $z=e^x=y+\sqrt{y^2-1}$ and $x=ln\left(y+\sqrt{y^2-1}\right)$ is the inverse function. **The inverse hyperbolic sine function** $y=arcchx=ln\left(x+\sqrt{x^2-1}\right)$ **has domain [1,+∞) and range [0,+∞) .**

The graphs of hyperbolic sine and cosine (bold) and inverse hyperbolic sine and cosine are presented on Figure 1.

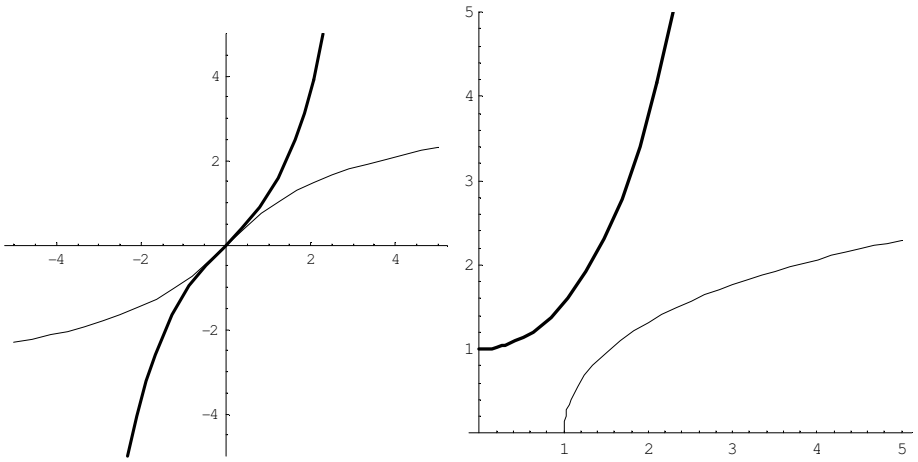

Fig. 1: The graphs of hyperbolic sine (left) and cosine (right, in bold) and inverse hyperbolic sine and cosine in [-5,5]x[-5,5] and [0,5]x[0,5].

The graphs are plotted by the Mathematica program:

```
g[1]=Plot[Cosh[x],{x,0,5},PlotStyle→{Thickness[0.008]},
PlotRange→{0,5}]
g[2]=Plot[ArcCosh[x],{x,1,5},PlotStyle→{Thickness[0.007]},
PlotRange→{0,5}]
f[1]=Show[g[1],g[2], AspectRatio→Automatic]
g[3]=Plot[Sinh[x],{x,-5,5},PlotStyle→{Thickness[0.008]},
PlotRange→{-5,5}]
g[4]=Plot[ArcSinh[x],{x,-5,5},PlotStyle→{Thickness[0.006]},
PlotRange→{-5,5}]
f[2]=Show[g[3],g[4], AspectRatio→Automatic]
```

3. The hyperbolic tangent function is $y = thx = \frac{shx}{chx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ **with domain R and range (-1,1).** The function is increasing and invertible.

The inverse hyperbolic tangent function is $y = arcthx = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$ **has domain (-1,1) and range R.**

4. The hyperbolic cotangent function is

$y = cthx = \frac{chx}{shx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ **with domain R\{0} and range R\[-1,1].** The function is decreasing and **the inverse hyperbolic cotangent function is**

$y = arc\,coth\,x = \frac{1}{2} ln\left(\frac{x+1}{x-1}\right)$ **with domain R\[-1,1]. and range R\{0}.**

The graphs of hyperbolic tangent and cotangent (bold) and their inverses are presented on Figure 2.The graphs are plotted by the program:

```
g[1]=Plot[Tanh[x],{x,-5,5},PlotStyle→{Thickness[0.008]},
PlotRange→{-5,5}]
g[2]=Plot[ArcTanh[x],
{x,-1,1},PlotStyle→{Thickness[0.006]},PlotRange→{-5,5}]
f[1]=Show[g[1],g[2], AspectRatio→Automatic]
g[3]=Plot[Coth[x],{x,-5,5},PlotStyle→{Thickness[0.008]},
PlotRange→{-5,5}]
g[4]=Plot[ArcCoth[x],
{x,-5,-1},PlotStyle→{Thickness[0.006]},PlotRange→{-5,5}]
g[5]=Plot[ArcCoth[x],{x,1,5},
PlotStyle→{Thickness[0.006]},PlotRange→{-5,5}]
g[6]=Show[g[4],g[5], AspectRatio→Automatic]
f[2]=Show[g[3],g[6], AspectRatio→Automatic]
```

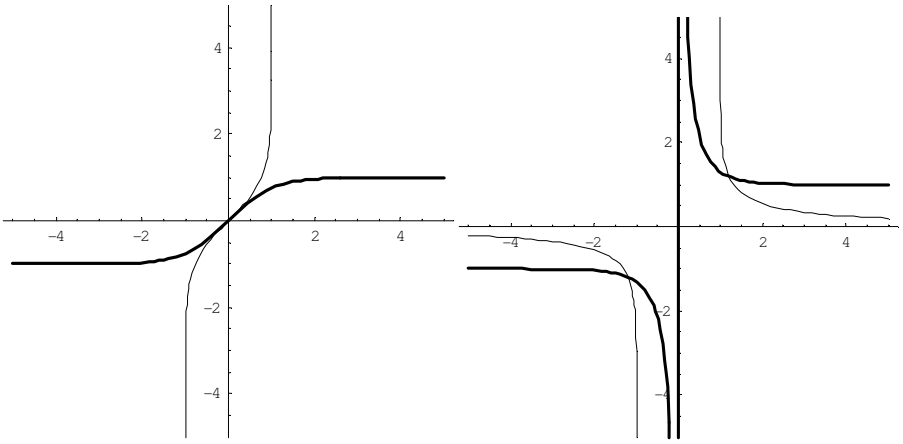

Fig. 2: The graphs of hyperbolic tangent and cotangent (bold) and their inverses in [-5,5]x[-5,5].

We have the following basic relations by the definitions

(4) $$ch^2 x - sh^2 x = 1, \quad x \in R$$

(5) $$arcth(x) = arccth\left(\frac{1}{x}\right), \quad |x| < 1.$$

Proposition 2. We have for $N \geq 2$

(6) $$N - 1 = \left(cth\left(arcth\left(sh\left(arccth\sqrt{N}\right)\right)\right)\right)^2.$$

Proof. By (4) and (5), for $x = shy \in (0,1)$ it follows

(7) $$arcth(shy) = arccth\left(\frac{1}{shy}\right), \quad 0 < shy < 1.$$

Denote $y = arccth\sqrt{N}$ for $N > 2$. We have

$$0 < y = arccth\sqrt{N} < arccth\sqrt{2} < arccth(1.4142) = \frac{1}{2} ln\left(\frac{24142}{4142}\right) = 0.88139 < 1$$

and

$$0 < x = shy < sh\left(arccth\sqrt{2}\right) = sh\left(\frac{1}{2} ln\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right)\right) = 1.$$

Then, by (4), (7) and $N = cth^2 y$, we obtain $\sqrt{N-1} = \sqrt{cth^2 y - 1} = \frac{1}{shy}$

and

$$arccth\left(\sqrt{N-1}\right) = arcth\left(sh\left(arccth\sqrt{N}\right)\right), \; N > 2,$$

which implies (6), i.e. $N - 1 = \left(cth\left(arcth\left(sh\left(arccth\sqrt{N}\right)\right)\right)\right)^2$, $N > 2$. Moreover

$\lim_{N \to 2} \left(cth\left(arcth\left(sh\left(arccth\sqrt{N}\right)\right)\right)\right)^2 = 1 = 2 - 1.$ ■

Combining Propositions 1 and 2, we obtain the curious

Corollary 3. *We have for x≥1*

(8) $$x = cth(arcth(sh(arccth(tg(arc\,cot\,g(cos(arctgx))))))),$$

(9) $$x = cth(arcth(sh(arccth(tg(arc\,cot\,g(sin(arc\,cot\,gx))))))),$$

and for x≥2

(10) $$x = tg(arc\,cot\,g(cos(arctg(cth(arcth(sh(arccthx))))))),$$

(11) $$x = tg(arc\,cot\,g(sin(arc\,cot\,g(cth(arcth(sh(arccthx))))))).$$

To express (N-1)/N with N, we obtain

Proposition 4. We have for $N \geq 1$

(12) $$\frac{N-1}{N} = \left(th\left(arccth\left(ch\left(arccth\sqrt{N}\right)\right)\right)\right)^2 .$$

We left Proposition 3 for a reader's exercise. It can be checked by *Mathematica* command

```
Simplify[Tanh[ArcCoth[Cosh[ArcCoth[Sqrt[x]]]]]]
```

The result is

$$\sqrt{\frac{-1+x}{x}} .$$

One can derive as well:

$$N-1 = \left(th\left(arccth\left(sh\left(arccth\sqrt{N}\right)\right)\right)\right)^2 ,$$

$$\frac{N}{N-1} = \left(cth\left(arccth\left(ch\left(arccth\sqrt{N}\right)\right)\right)\right)^2 .$$

3. BRAHISTOHRONE PROBLEM AND ORIGIN OF CALCULUS OF VARIATIONS

The brachistochrone problem (in Greek: βράχιστος, *brachistos* means "the shortest", and χρόνος, *chronos* – "time") is one of the earliest problems in the Calculus of variations. The problem is to find the curve C, joining two points $P_a(a,A)$ and $P_b(b,B)$ on the plane with a<b, A>B, on which a point-like body passed from $P_a(a,A)$ to $P_b(b,B)$, accelerated by gravity, for a least amount of time.

Johann Bernoulli [1667-1748] posed the problem of the brachistochrone to the readers of *Acta Eruditorum* in June, 1696. The journal *Acta Eruditorum* was founded in 1682 in Leipzig by Otto Mencke [1644–1707] and Gottfried Leibniz [1646-1727]. Six months were allowed by Bernoulli for the solution of the problem, and in the event of none being sent to him he promised to publish his own. Bernoulli received a letter from Leibniz, and requesting that the period for their solution should be extended to Christmas next.

Five mathematicians responded with solutions: Isaac Newton [1642-1727], Jakob Bernoulli [1654-1705] (Johann's brother), Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus [1651 –1708] and Guillaume de l'Hôpital [1646-1727].

On 29 January 1697 Newton returned at 4 p.m. from working at the Royal Mint and found in his post the problems that Bernoulli had sent to him directly;

two copies of the printed paper containing the problems. Newton stayed up to 4 a.m. before arriving at the solutions; on the following day he sent a solution to Montague. The solution was published, unsigned in “Philosophical Transactions”, May, 1697. Although Newton's solution was anonymous, Johan Bernoulli recognized the author and said

"tanquam ex ungue leonem" (we recognize the lion by his claw).

The brachistochrone problem is historically important because it initiate the branch of mathematics now known as the Calculus of Variations. It can be fixed classical, modern and post-modern periods of it’s development (see [5], [7]).

The problem of the Calculus of variations is to find a function of given class, minimizing a given integral (functional) depending on the functions of this class. The approach was developed by Leonard Euler [1707-1783] in 1736 and Joseph-Lois Lagrange [1736-1813] in 1755. It deals with general variational problems of following type:

Given a function (Lagrangian) of three variables F(x, y, p), to find a function $y \in C^1([a,b]), y(a) = A, y(b) = B$ for which the integral (energy functional) $E(y) = \int_a^b F(x, y(x), y'(x))dx$ attends a minimim value. The minimizing function satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

with boundary conditions $y \in C^1([a,b]), y(a) = A, y(b) = B$.

The brachistochrone problem is of this form and for it

$$F(x, y(x), y'(x)) = \sqrt{\frac{1 + y'^2(x)}{2gy(x)}}.$$

The Euler-Lagrange equation reduces to

$$F - y'\frac{\partial F}{\partial y'} = const, \tag{12}$$

because F does not depend on x (see [8]). For simplicity, for boundary conditions, we can suppose that a=A=0, b>0, B<0. By simple computations (12) reduces to

$$y(x)\left(1 + y'^2(x)\right) = 2C,$$

which solution in parametric form is the cycloid

$$\begin{cases} x = C(t - \sin t) + K, \\ y = C(1 - \cos t). \end{cases}$$

In order to satisfy the boundary conditions when t=0, x=y=0, we have K=0 and to find C we ask for the unique solution $t_0<0$ of the equation

$0 > \frac{B}{b} = \frac{1-\cos t}{t-\sin t}$. Then $C = \frac{b}{t_0 - \sin t_0}$ and $x(t_0) = b, y(t_0) = B$.

For the general conditions a<b, A>B, the solution is an arc of a cycloid joining the points $P_a(a,A)$ and $P_b(b,B)$:

$$. \begin{cases} x = a + C(t - \sin t), \\ y = A + C(1 - \cos t),. \end{cases}$$

where $C = \frac{b-a}{t_1 - \sin t_1}$ and $t_1<0$ is the unique solution of the equation

(13) $$0 > \frac{B-A}{b-a} = \frac{1-\cos t}{t-\sin t}.$$

This allows us to plot the cycloids, using parametric plot commands in Mathematica. On Figure 3 are presented three cycloids, joining the points (0,1) and (2,0), (0,2) and (2,0) and third one joining (0,1) and (3,0). For the first one, the solution of (13) is found by

$$FindRoot\left[\frac{1-Cos[x]}{x-Sin[x]} == -0.5, \{x, -0.1\}\right]$$

and is x_0= -3.50837. Then

$$C = N[\frac{2}{x_0 - Sin[x_0]}, 4] = -0.5172.$$

The first cycloid is plotted by the command

```
f1=ParametricPlot[{-0.5172(t-Sin[t]),1-0.5172(1-Cos[t])},
{t,-3.5083,0},AspectRatio→Automatic, PlotStyle→{Thickness[0.01]}]
```

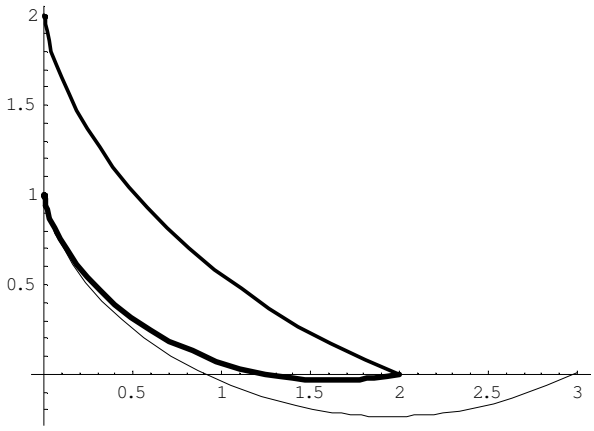

Fig 3: Cycloids joining points (0,1) and (2,0); (0,2) and (2,0); (0,1) and (3,0).

4. EXAMPLES ON FRACTIONAL DIFFERENTIAL EQUATIONS

Fractional calculus and fractional differential equations are generalizations of the usual calculus and differential equations to a non-integer order differentiation and integration. The subject is old and goes back to XVII century, when Gotfried Leibniz and Isaak Newton invented the differential calculus. In a letter to L`Hôpital in 1695 Leibniz stated the following question *"Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?"* L`Hôpital was curious about the question and replied by another question to Leibniz *"What if the order will be 1/2?"* Many known mathematicians contributed to theory of farctional calculus over 300 years, among them Liouville, Riemann, Weyl, Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov (see [7]). Recently, a great attention has been focused on the study of boundary value problems (BVP) for fractional differential equations. They appear in mathematical models in different branches in Science as Physics, Chemistry, Biology, Geology, as well as, Control theory, Signal theory, Nanoscience and so on. The existence and multiplicity of solutions for BVP for nonlinear fractional differential equations is extensively studied using various tools of nonlinear analysis as fixed point theorems, degree theory and the method of upper and lower solutions (M. Belmekki, J.J.Nieto, R. Rodriguez-Lopez [1], M. Benchohra, A. Cabada, D. Seba [2]). Starting with the pioneering work of Jiao and Zhou [4], the variational methods are applied to fractional differential equations. Recall the left and right Riemann-Liouville and Caputo fractional derivatives (see [6]) for $f \in AC([a,b])$ and $0 < \alpha \leq 1$ as follows:

$$
\begin{aligned}
{}_aD_t^{\alpha} f(t) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_a^t \frac{f(s)ds}{(t-s)^{\alpha}} \right), \\
{}_tD_b^{\alpha} f(t) &:= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_t^b \frac{f(s)ds}{(t-s)^{\alpha}} \right), \\
{}_a^cD_t^{\alpha} f(t) &:= {}_aD_t^{\alpha-1} f'(t) \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_a^t \frac{f'(s)ds}{(t-s)^{\alpha}} \right), \\
{}_t^cD_b^{\alpha} f(t) &:= -{}_tD_b^{\alpha-1} f'(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_t^b \frac{f'(s)ds}{(t-s)^{\alpha}} \right).
\end{aligned}
$$

Note that when $\alpha = 1, {}_a^cD_t^1 f(t) = f'(t), {}_t^cD_b^1 f(t) = -f'(t)$ In [3], we consider the Dirichlet's boundary value problem for fractional differential equation with impulsive effects:

$$
\text{(P)} \quad \begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha} u(t)\right) + a(t)u(t) = \lambda f(t, u(t)), \\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha} u\right)\right)(t_j) = \mu I_j\left(u(t_j)\right) \quad , \\ u(0) = u(T) = 0, \end{cases}
$$

where $1/2 < \alpha \le 1, \lambda > 0, \mu > 0; 0 < t_j < T, j = 1,...n$ and

$$
\Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha} u\right)\right)(t_j) = {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha} u\right)(t_j^+) - {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha} u\right)(t_j^-).
$$

Using the variational method and three critical point theorem it is proved in [3]. the following result

Corollary 5*. Suppose that* $1/2 < \alpha \le 1$ *and let f:***R**$\to$**R** *be a nonnegative continuous function with* $f(1) \ne 1$ *and* $a \in C([0,T])$ *with* $a_0 = \inf\{a(t) : t \in [0,T]\} > 0$ *. Assume also that*

$$
\lim_{|x| \to \infty} \frac{f(x)}{x} = \lim_{|x \to 0^+} \frac{f(x)}{x} = 0.
$$

Then, for each $\lambda > \dfrac{2K}{\dfrac{1}{T}\displaystyle\int_0 f(s)ds}$*, where*

$$K=\frac{T}{3}\,\|a\|_{\infty}+\frac{6\alpha^2-19\alpha+16}{2\Gamma^2(1-\alpha)(1-\alpha)^2(2-\alpha)(3-2\alpha)}\left(\frac{T}{4}\right)^{1-2\alpha},\quad 1/2<\alpha\le 1,$$

and for each nonnegative continuous functions such that

$$\lim_{|x|\to\infty}\frac{I_j(x)}{x}=\lim_{|x\to 0^+}\frac{I_j(x)}{x}=0,\ j=1,\dots,n,$$

the problem

$$(P_1)\begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)+a(t)u(t)=\lambda f(t,u(t)),\quad t\neq t_j,\ j=1,\dots,n,t\in(0,T),\\ -\Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_j)=I_j\left(u(t_j)\right),\quad j=1,\dots,n,\\ u(0)=u(T)=0,\end{cases}$$

admits at least three classical solutions.

One can give an example as an application, where calculations and plots with Mathematica are used.

Example 6. Let T=a=n=1 and $t_1\in(0,1)$. Consider the problem

$$(Q)\begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)+a(t)u(t)=24\min\{|u(t))|^2,\sqrt{|u(t))|}\},\quad t\in(0,1)\setminus\{t_1\},\\ -\Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_1)=\min\{|u(t_1))|^2,\sqrt{|u(t_1)|}\},\\ u(0)=u(1)=0,\end{cases}$$

We look for α for which, the inequality $24>\dfrac{2K}{\int_0^1 s^2\,ds}=6K$ holds, that is

4>K=*h(α)*, where

$$K=h(\alpha)=\frac{1}{3}+\frac{6\alpha^2-19\alpha+16}{2\Gamma^2(1-\alpha)(1-\alpha)^2(3-2\alpha)(2-\alpha)4^{1-2\alpha}},\quad 1/2<\alpha\le 1.$$

The function h(α) is increasing and the unique root of the equation 4=h(α) is α_1 =0.789328. The graphs of the constant function 4 and function *h* are presented on Figure 4.

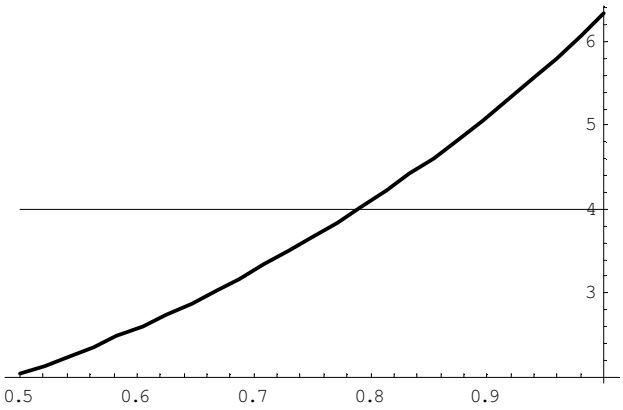

Fig 4. The graphs of the constant function 4 and function h for $\alpha \in (1/2,1]$.

We can take α in the interval (0.5; 0.7893). The assumptions of Corollary 4 are satisfied and the problem (Q) has at least three classical solutions.

The constant 24 in the equation of (Q) can be replaced by a constant M,such that 0<M<38. In this case the problem (Q) has at least three solutions provided that $\alpha \in (0.5,\alpha_0)$, where $\alpha_0 \in (0.5,1)$ is the unique root of the equation

$$\frac{M}{6} = \frac{1}{3} + \frac{6\alpha^2 - 19\alpha + 16}{2\Gamma^2(1-\alpha)(1-\alpha)^2(3-2\alpha)(2-\alpha)4^{1-2\alpha}}, \quad 1/2 < \alpha < 1.$$

It follows by the limit computation $\lim_{x\to 0} x\Gamma(x) = 1$ and

$$\lim_{\alpha\to 1}[\frac{1}{3} + \frac{6\alpha^2 - 19\alpha + 16}{2(3-2\alpha)(2-\alpha)4^{1-2\alpha}}] = \frac{19}{3} = 6.333...$$

and 38/6=19/3=6.333… .

5. CONCLUSIONS

Three examples for solving and investigating mathematical problems using computer algebra systems Mathematica and Maple. They give a ground for research studies on various Calculus problems. They show a knowledge and skills in Mathematics and Informatics in solving of research problems. The teachers can show and formulate to the students appropriate and curious problems opening the roads to research studies and innovations.

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Place of the Logical Symbolism for Revealing the Logical Structure of the Definitions of Some Mathematical Notions

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Abstract. *The article discusses the application of formal means for revealing the logical structure of the definitions of some mathematical notions.*

Keywords: *logical structure of the definition, absolute value of a real number*

1. INTRODUCTION

The combination of the formal-logical presentation of information and the interpretation of its contents are the basis in the training of the future teachers in mathematics. The presentation of the formal-logical basis of the respective didactic task relates to the training of the University students at a macro-level ([7], [3]). At that level formal models are used for revealing the essence of the discussed private-didactic task of the theoretical and professional training of the teachers. The logical apparatus is used in two aspects. Fist, in order to reveal the logical structure of the knowledge for the purpose of clarifying the complexity of the structure of knowledge. Second, the training of the future teachers is raised to higher level due to the fact that it does not depend on the concrete contents, but on its structure. That on its behalf raises the effectiveness of the training. Thus these means perform a prognostic-structural function in the solution of didactic problems related with the qualification of the teachers.

The logical component directly relating to the studying of the subject component (as per the terminology [4]) of the educational contents,

a) assists the correct defining of the reasoning structure;

b) provides the opportunity for carrying out control on one's own or other person's reasoning and arguments;

c) provides the opportunity for establishing the reasons for the difficulties the students meet;

d) represents part of the basis for formation of an action plan;

e) represents part of the basis for re-formulation of statements / problems;

f) has an impact on the heuristic capacities of the teacher by means of specification of methods, forms and means for realization of the respective contents before the students;

g) represents part of the basis for the specification of the adequate activities of the teacher.

The logical component not only shows the way of organizing the subject component, but it is also an instrument for the transformation of the logical structure of the statements. As a result of these transformations new logical structures arise, the contents interpretation of which provides new mathematical facts or a new interpretation of already known facts and brings about a change in the subject component, i. e. a dynamic link between the two components of the educational contents is at hand.

The contents interpretation of the discussed models brings about the setting up of didactic technologies, on the basis of which later through specifications of the activities and the means these contents are presented to the students. That represents part of the training of the teachers at a mezzo-level ([7], [3]).

At the micro-level ([7], [3]) the knowledge is made specific and concrete in the form of problems, systems of problems or didactic situations used as an effective didactic instrument for training the students. The ways and methods for the solution of these problems depend on the knowledge, on the capacities of the students, on the professional and theoretical qualification of the teacher.

In the further part of this presentation we shall concentrate on the application of the logical symbolism for modeling the definitions of some notions. The logical symbolism plays a major part in expressing the structure of knowledge, which structure can later be transformed on the basis of the laws of logics. On the basis of the formal model we shall make some recommendations with regard to the studying of these notions in the school course in mathematics.

2. APPLICATIONS

2.1. Examples 1 - definition of the notion „absolute value of a real number"

2.1.1. Theoretical bases

Encyclopedia of Mathematics (1977), Moscow, Russia, Soviet Encyclopedia, 1, 42.

The absolute meaning of a skew-field (non-commutative field) is called mapping φ of the skew-field K in the set $\mathbb{R}$ of the real numbers fulfilling the conditions:

1) $\varphi(x) \geq 0$ и $\varphi(x) = 0 \Leftrightarrow x = 0$

2) $\varphi(x.y) = \varphi(x).\varphi(y)$

3) $\varphi(x+y) \leq \varphi(x) + \varphi(y).$

The absolute meaning is often denoted by $|x|$ instead of $\varphi(x)$. The absolute meaning is also called norm.

If $K = \mathbb{R}$, then $|x| = max\{x, -x\}$ is called absolute value or module of $x \in \mathbb{R}$.

If $K = \mathbb{C}$, then $|x| = \sqrt{x.\bar{x}}$.

Proposition. ([6], page 81.)

$(p \to r) \wedge (q \to r) \Leftrightarrow p \vee q \to r$

Proof.

$(p \to r) \wedge (q \to r) \Leftrightarrow (\bar{p} \vee r) \wedge (\bar{q} \vee r) \Leftrightarrow (\bar{p} \wedge \bar{q}) \vee r \Leftrightarrow (\overline{p \vee q}) \vee r \Leftrightarrow p \vee q \to r.$

In the proof of this proposition is used the equivalence $p \to q \Leftrightarrow \bar{p} \vee q$ and the fact that the disjunction distributes over the conjunction.

It has been proven in [6] on page 177 that $\overset{n}{\underset{i=1}{\vee}} p_i \to q \Leftrightarrow \overset{n}{\underset{i=1}{\wedge}} \bar{p_i} \vee q \Leftrightarrow \overset{n}{\underset{i=1}{\wedge}} (\overline{p_i} \vee q) \Leftrightarrow \overset{n}{\underset{i=1}{\wedge}} (p_i \to q)$.

It follows from that equivalence that the conjunction of the *n* number theorems with structure $p_1 \to q, p_2 \to q, ..., p_n \to q$ is equivalent to the proposition with structure $\overset{n}{\underset{i=1}{\vee}} p_i \to q$.

On the basis of that proposition and its generalization we can present the formal logical structures of the definitions of some mathematical notions.

2.1.2. Practical realization

According to the programs in mathematics currently used in Bulgaria the notion absolute value of a rational number is introduced in the 6-th class. In the educational documentation that notion is introduced as unary operation in the set of the rational numbers, i.e. as a mapping of the kind $f : \mathbb{Q} \to \mathbb{Q}^+$, which relation is presented by text or symbolically.

Some nearer specifications are necessary for the understanding of the logical relation between the characteristic features of this relation and the logical loading of the symbolism used.

In [1] the definition of the notion absolute value of a rational number has been symbolically written in the following way:

$$|x| = y \Leftrightarrow \begin{cases} if \;\; x \geq 0 \,, \; then \;\; y{=}x \\ if \;\; x{<}0 \,, \; then \;\; y{=}{-}x \end{cases} \qquad (1)$$

or $$y = |x| \Leftrightarrow (x \geq 0 \to y = x) \wedge (x < 0 \to y = -x) \;. \qquad (2)$$

The set of points, which satisfies the condition

$$x_0 \geq 0 \rightarrow x_0 = y_0 , \quad (3)$$

are all points from the left semi-plane with respect to the axis Oy and the points from the bisector of 1st quadrant (Fig.1.), since the equivalence of the following assertions holds

$$x_0 \geq 0 \rightarrow x_0 = y_0 \Leftrightarrow \overline{x_0 \geq 0} \vee x_0 = y_0 \Leftrightarrow x_0 < 0 \vee x_0 = y_0 .$$

If $x_0 < 0$, then the condition (3) has been fulfilled, because the premise of the material implication is false.

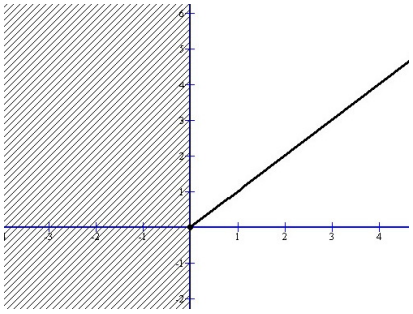

Fig.1.

The set of points satisfying the condition

$$x_0 < 0 \rightarrow y_0 = -x_0 , \quad (4)$$

are all points from the right semi-plane with respect to the axis Oy and the points from the bisector of the 2nd quadrant (Fig.2.), since the equivalence of the following proposition holds

$$x_0 < 0 \rightarrow y_0 = -x_0 \Leftrightarrow \overline{x_0 < 0} \vee y_0 = -x_0 \Leftrightarrow x_0 \geq 0 \vee y_0 = -x_0 .$$

If $x_0 \geq 0$, then the condition (4) has been fulfilled, because the premise of the material implication is false.

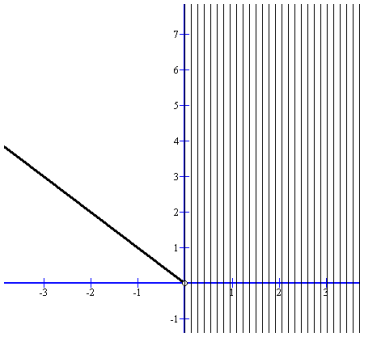

Fig.2.

The intersection of the two sets is the graph of the function $y = |x|$ (Fig.3.).

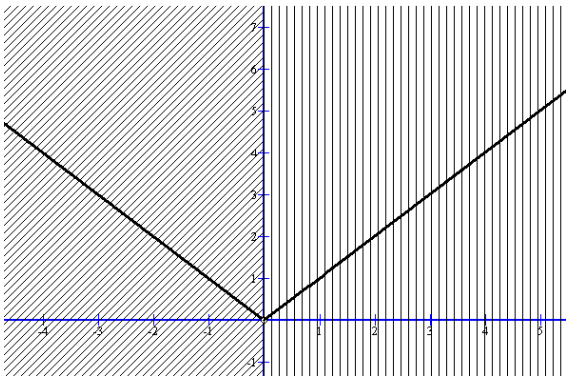

Fig.3.

Consequently, the right side of the equivalence (2) leads to obtaining true statements. That means that the connection between the two statements is conjunctive, which symbolically is written down by a curly bracket or by a vertical line.

The curly bracket is used also in the writing down of an analytical kind of a function presentation, set by different analytical expressions in the different definition intervals.

In the symbolical writing down of the definitions of some notions, whose logical structure of the defining part is complex (conjunctive, disjunctive or more complex), curly brackets (vertical line) or square bracket can be used as conventionally accepted symbols, replacing the respective logical operations.

If we accept the following denotations for the assertions from (2)

$$y=|x| \Leftrightarrow \left(\underbrace{x \geq 0}_{p} \rightarrow \underbrace{y=x}_{q} \right) \wedge \left(\underbrace{x<0}_{\overline{p}} \rightarrow \underbrace{y=-x}_{r} \right),$$

then by the means of the reasoning calculation we can prove that

$$(p \rightarrow q) \wedge (\overline{p} \rightarrow r) \Leftrightarrow (p \wedge q) \vee (\overline{p} \wedge r),$$

since in such case $v(q \wedge r)=0$. That equivalence shows that the right side of the equivalence (2) can be written down in the form

$(x \geq 0 \wedge y=x) \vee (x<0 \wedge y=-x)$ or $y=|x| \Leftrightarrow (x \geq 0 \wedge y=x) \vee (x<0 \wedge y=-x)$ (5)

or $y=|x|=\left[\begin{array}{l} |x<0 \\ |y=-x \\ |x \geq 0 \\ |y=x \end{array}\right.$. (6)

That last taking down is more complicated and therefore traditionally one of the following symbols recordings ((7) or (8)) are used of the definition of

the notion absolute value of a real number, which written expressions are more concise and more compact -

$$|a| = \begin{cases} a, & a > 0 \\ 0, & a = 0 \\ -a, & a < 0 \end{cases} \quad (7)$$

or $|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$. (8)

That on its part raises the question: Are these written expressions equivalent?

On the basis of the given proposition it follows that

$$\underbrace{a > 0}_{p} \to \underbrace{|a| = a}_{r} \quad \wedge \quad \underbrace{a = 0}_{p} \to \underbrace{|a| = 0}_{r} \quad \Leftrightarrow \underbrace{a \geq 0}_{p \vee q} \to \underbrace{|a| = a}_{r}. \quad (9)$$

As per (9) and the associativity of the operation conjunction it follows that the two definitions ((7) and (8)) are equivalent, because

$$\left(\underbrace{a > 0}_{p} \to \underbrace{|a| = a}_{r} \right) \wedge \left(\underbrace{a = 0}_{q} \to \underbrace{|a| = 0}_{r} \right) \wedge \left(\underbrace{a < 0 \to |a| = -a}_{s} \right) \Leftrightarrow$$

$$\left(\left(\underbrace{a > 0}_{p} \to \underbrace{|a| = a}_{r} \right) \wedge \left(\underbrace{a = 0}_{q} \to \underbrace{|a| = 0}_{r} \right) \right) \wedge \left(\underbrace{a < 0 \to |a| = -a}_{s} \right) \Leftrightarrow$$

$$\left(\underbrace{a > 0}_{p} \vee \underbrace{a = 0}_{q} \to \underbrace{|a| = a}_{r} \right) \wedge \left(\underbrace{a < 0 \to |a| = -a}_{s} \right) \Leftrightarrow \left(\underbrace{a \geq 0}_{p \vee q} \to \underbrace{|a| = a}_{r} \right) \wedge \left(\underbrace{a < 0 \to |a| = -a}_{s} \right).$$

2.1.3. Practical conclusions

Notwithstanding the fact that the notions positive number, non-negative number, zero, opposite numbers are preliminary known to the students, they meet difficulties in the learning of the notion absolute value of a real number. Its structure also influences the learning of the definition.

The complex logical structure of the defining part of the definition of a notion absolute value of a real number explains part of the difficulties of the students when learning and assimilation of that notion.

Another reason for the difficulties for the students is due to the fact that the notion is defined by mapping. Until that moment (6-th class) the students have studied only binary operations. That represents their first meeting with an unary operation, which has not been prepared in advance.

Exercises for the learning and assimilation of the separate parts of the definition and the logical relations between them are not to be found in the textbooks. Therefore the learning of some notions with complex structures of the definition causes difficulties and mistakes on the part of the students.

The logical structure of the definition determines also the structure of the activities for its learning. Therefore the process of recognition of the mathematical objects on the basis of the definitions of the mathematical notions requires a good understanding of the structure of the definition of the notion to enable the description of the adequate activities of the teacher in a respective technology following the said structure.

2.2. Other examples

Analogical logical structures have also definitions of the following notions.

2.2.1. Example 2 - definition of the notion „scalar product"

That is a mapping of the kind $V \times V \to \mathbb{R}$, where V is a set of vectors. That means that each ordered pair of the kind $\left(\vec{v},\vec{w}\right), \vec{v} \in V, \vec{w} \in V$ must have an image in $\mathbb{R}$.

The logical structure of the definition of the operation „scalar product" is as follows

$\vec{a}\vec{b} = \begin{cases} \left|\vec{a}\right|\left|\vec{b}\right| cos\angle\left(\vec{a},\vec{b}\right), & \vec{a} \neq \vec{0} \wedge \vec{b} \neq \vec{0} \\ 0, & \vec{a} = \vec{0} \vee \vec{b} = \vec{0} \end{cases}$, or when written down without the curly bracket the structure of the definition is

$\left(\vec{a} \neq \vec{0} \wedge \vec{b} \neq \vec{0} \to \vec{a}\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| cos\angle\left(\vec{a},\vec{b}\right)\right) \quad \wedge \quad \left(\vec{a} = \vec{0} \vee \vec{b} = \vec{0} \to \vec{a}\vec{b} = 0\right).$

2.2.2. Example 3 – definition of the notion „scalar multiplication"

That is a mapping of the kind $\mathbb{R} \times V \to V$. That means that each ordered pair of the kind $\left(\lambda,\vec{v}\right), \lambda \in \mathbb{R}, \vec{v} \in V$ must have an image in V.

The logical structure of the definition of the operation „ scalar multiplication" is as follows:

$$\lambda\vec{a} = \vec{b} = \begin{cases} \vec{0}, \lambda = 0 \vee \vec{a} = \vec{0} \\ \left|\vec{b}\right| = \lambda\left|\vec{a}\right| \wedge \vec{b} \uparrow\uparrow \vec{a}, \quad \vec{a} \neq \vec{0} \wedge \lambda > 0 \\ \left|\vec{b}\right| = |\lambda|\left|\vec{a}\right| \wedge \vec{b} \uparrow\downarrow \vec{a}, \quad \vec{a} \neq \vec{0} \wedge \lambda < 0 \end{cases},$$

or when written down without using the curly bracket the structure of the definition is

$\left(\lambda = 0 \vee \vec{a} = \vec{0} \to \lambda\vec{a} = \vec{b} = \vec{0}\right) \quad \wedge \quad \left(\vec{a} \neq \vec{0} \wedge \lambda > 0 \to \left|\vec{b}\right| = \lambda\left|\vec{a}\right| \wedge \vec{b} \uparrow\uparrow \vec{a}\right) \quad \wedge$

$\left(\vec{a} \neq \vec{0} \wedge \lambda < 0 \to \left|\vec{b}\right| = |\lambda|\left|\vec{a}\right| \wedge \vec{b} \uparrow\downarrow \vec{a}\right).$

The logical structures of these three definitions are the same. That means also the use of one common approach in their introduction and learning, i.e. in the process of their studying. The activities are realized on concrete contents, but their structure does not depend on the concrete contents.

3. CONCLUSIONS

The formal-logical knowledge can be used as means for modeling of definitions or for the assessment of formulations of definitions.

The meaning of the symbolic expression of the definition of one notion consists in the fact that symbolism helps to express and makes understandable the logical structure of the characteristic features of the definition, but also for expressing the structure of the definition itself [2]. It is only at this point that we realize how complex that structure is, what thinking activity the students have to carry out when working with that definition. The internal structure of the definition determines also different ways of action in its application.

On these grounds the following recommendations become necessary:

- to carry out analysis of the logical structure of the characteristic features of the definition of a notion at its introduction;
- on the basis of the logical structure of the definition of a notion to generate tasks for its practicing;
- for the purpose of facilitating the understanding and learning of the definition it is to the point to bring the separate part of the definition to sensually acquired concretizations or interpretations, because such definitions exist, which associate images in the conscience of people but there exist also such definitions that combine facts accessible only for the intellect, and which in their abstractedness are deprived of any material contents [5]. Thus we reach a perfect purity for the sake of drifting far from reality.

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Visualizations of Algorithms and Software Programs

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***Abstract**: In this article has been made a classification of the means for software visualizations. Their applications in the teaching of computer programming are pointed. Methods for automatically visualization of program code are discussed.*

***Keywords:** visualization, algorithms, means for visualizing*

1. INTRODUCTION

The visualizations are a powerful and necessary tool in the scientific research, a means of presenting complex situations in the program code on an intuitive level. They are of a great practical importance for the visual display of the information from different fields. According to the field of application the visualizations are classified into the following groups:

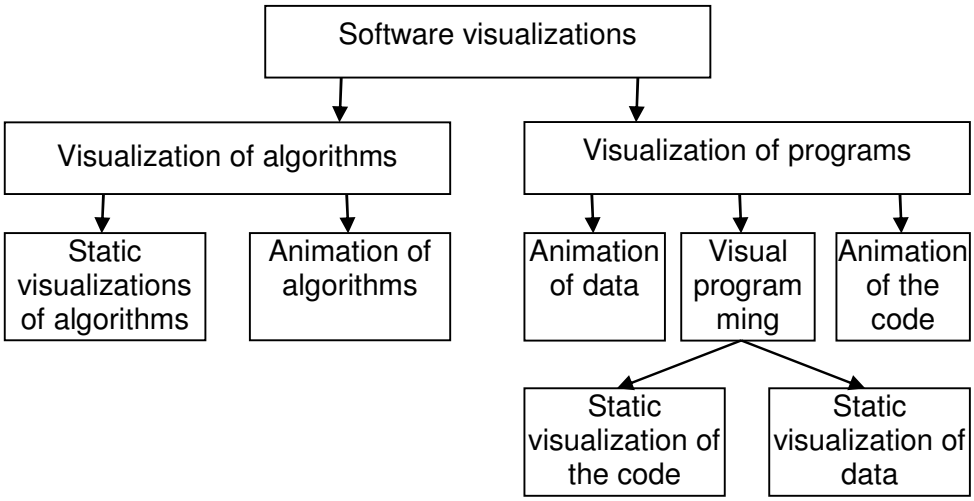
- scientific visualizations
- software visualizations
- visualizations of information [3]

The article provides an overview of the software visualizations and researches their applications in the teaching of computer programming.

2. SOFTWARE VISUALIZATIONS IN THE TEACHING OF COMPUTER PROGRAMMING

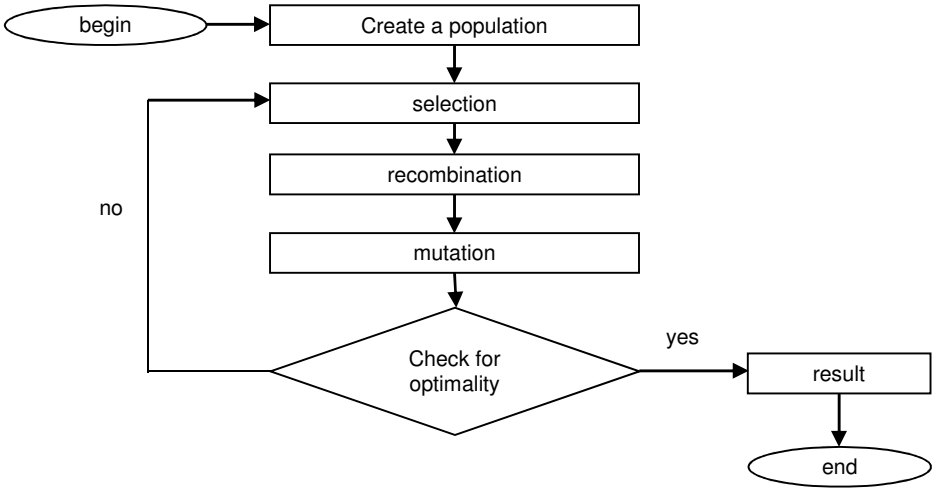
In the methodology of teaching the using of illustrations visual aids for an assimilation of the learning content allows accelerate the perception, the understanding and the analyzing of the results. Visual means can be employed for each of the three stages of the development of a software program - design, decision and evaluation of the results. They make easier better to be understood the work of the algorithm as trace its move and examine the interim results. They help to be evaluated the decision and to be seen the possible ways to its improving. The visual aids for representation of the data and the various algorithms significantly lighten the learning process.

Classification of the resources for software visualizations:



3. VISUALIZATION OF CLASSIC ALGORITHMS

Static visualizations usually are used with educational purpose for classic algorithms in design stage. The means of their creation are block diagrams. They are used for short algorithms or parts of the program code. For example, for genetic algorithm can build the following block diagram. With this algorithm evolutionarily can be solved complicated tasks.



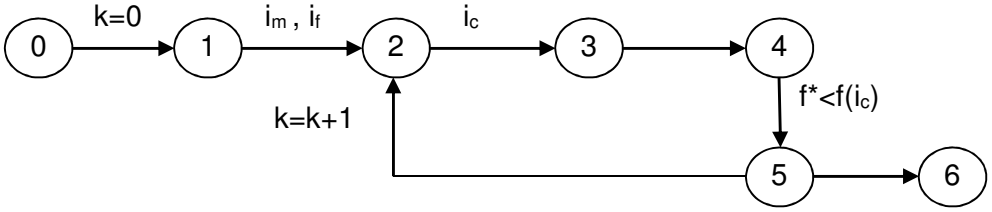
The animations of the algorithms are developed for the first time in 1980 as a way to abstract graphical representation of parts of the source code. One of the methods for animation of the algorithms is based on the separation of the events that are created during the execution of the algorithm. For each

event is realized a visual effect. The main stages of designing visualizer are: construction of a finite state machine; selection of variables for visualization; the image is formed by number of state and a value of visualized variables. A finite state machine is constructed, so that on each its state to match an event of the source algorithm. The procedural program is transformed into a program based on machine. [2]

For this genetic algorithm several events are considered

1. Select of an initial population I_0 and replace target function $f^*=\max\{f(i) , i\epsilon I_0 \}$, $k=0$.
2. From population I_k select parents i_m and i_f.
3. From i_m and i_f create i_c.
4. Modify i_c.
5. If $f^*<f(i_c)$, than $f^*= f(i_c)$.
6. Update the population and $\kappa=k+1$.

Based on these events a Mealy machine is created with seven states.



Another method of visualization is based on the change of the values of the variables in the memory. In the simple case, the values of the variables are arranged in a table. The steps of the algorithm, the values of the variables, the actions and the results of the actions are recorded in the following table.

step	κ	f*	f(i)	i_m	i_f	i_c	action	Check of a condition	result of check of a condition (0 or 1)

3.1. Designing of an environment for a visualizing

Visual representation of the states of the algorithm is implemented with a set of static or dynamic images. The graphical modeling language UML has appropriate collection of graphical notations for the selected conditions. The machine describing the algorithm is converted into machine for visualization. We can create an environment for visualizing with the following components: logic of the visualizer, data model, visual representation, management system and interface. Disseminated environments for visualizing are BALSA, TANGO, Leonardo and CATAI. They are supplemented by special programs (scripts) written on different programming languages. The idea is the animation process to be automated.

4. VISUALIZING DATA STRUCTURES AND ALGORITHMS

The systems for visualizing allow the algorithm to be implemented on steps and to be monitored the behavior of the computer program with different input data. Another important function is the animation of data structures - array, string, list, stack, queue, graph, binary tree. They have a complex structure and large volume. This makes difficult to analyze them in text form.

Effective tool for more qualitative understanding of the programming code is the visual programming. Dragon is a visual language, which uses two types of elements: graphic figures and text captions. The text captions are in the graphic forms or outside them. It describes data structures and designs programs. It has two syntaxes. The visual syntax specifies the rules for placement and connection of the geometric figures with connecting lines. The text syntax specifies the rules for combination and connection of symbols with geometric figures. The idea of using a visual language is to achieve a natural transition from visual description of the algorithm to visual programming.

5. CONCLUSIONS

The visual methods for description reduce the level of abstraction in the presentation of the algorithms. The combination of static and dynamic visualizations of the studied data structures and algorithms are useful in the teaching of computer programming. The application of visual forms allows to accelerate the perception of the decision to understand the reason for its occurrence to see possible ways to improve it.

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Test System and Software for Evaluation of Students' Knowledge of Programming

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***Abstract**: The article describes a test system for assessing students' knowledge of programming. It combines the evaluation with elements of the training process and inquiry-based approach in education. Software for the generation, verification and analysis of tests is also presented.*

***Keywords:** Multiple-choice test, software.*

1. INTRODUCTION

The idea of making a test system for assessing students' knowledge in programming arose in 1998, when I started to teach programming in Pascal to students from South-West University "Neofit Rilski" [4]. Then I wrote the first version of test generator. Since then I use this system in my courses in programming, object-oriented programming and data structures. The test system is also applied to courses in operating systems and advanced programming.

In terms of students, the test consists of three stages: preparation for the test, making the test, and verifying the results. The teacher must create a test, generate the required number of copies for the students, check the completed tests and analyze the results of the students.

2. SPECIAL MULTIPLE CHOICE TEST

The test presented here is of a multiple-choice type with some peculiarities with respect to the classic multiple-choice. It is on paper and is held within 1.5 astronomical hours.

2.1. Terminology

The test consists of items. Stem is the introductory question or statement at the beginning of each item. The stem is followed by the options. The options are: answers – the correct options, and distractors – the incorrect options. The items are stored in an item bank. Individual test consists of fixed number of items with fixed number of options.

2.2. Description

The individual tests are generated from the test bank randomly. The number of options in any individual test is fixed on 4 for all items. In any individual test the number of correct options (answers) of an item may be any number in the interval [0,4]. The student has to identify each answer and each distractor. For any option he/she has 3 choices of response:

- yes, i.e. I know the option is an answer;
- no, i.e. I know the option is a distractor;
- nothing, i.e. I do not know whether the option is an answer or a distractor.

Any correct response (yes or no) adds one point in the total score but any incorrect (opposite) response subtracts one point (a penalty point) from the total score.

We estimate an individual test of M items with x points total score as follows:

```
int p = ceil(100.0*x/(M*4)), e = 2;
     if (p >= 90) e = 6;
else if (p >= 76) e = 5;
else if (p >= 60) e = 4;
else if (p >= 50) e = 3;
```

At least one week before the date of the test all original stems and two example options per item (an answer and a distractor) are published online on the course website. A few days before the test day, a seminar is conducted where students discuss the published items and options, as well as other possible proposals for options.

During the completion of the test the students can use lectures, textbooks and any other printed materials. They are allowed also to use computer as a book, or also compiler, or even Internet. Anyone can ask questions about ambiguities in the test.

After the verification of the tests the individual tests and responses are returned to the students. Each student should carefully check his/her individual test in order to determine whether he/she agrees with the noted errors. If something is not clear she/he can discuss the case. It is normal to increase the total score of the student if his/her arguments about the case are reasonable.

Arguments for this way of doing the test can be found at [1] and [2].

2.3. Inquiry-Based Approach

According to a definition by Linn, Davis and Bell [3], inquiry is the intentional process of diagnosing problems, critiquing experiments, distinguishing alternatives, planning investigations, researching conjectures,

searching for information, constructing models, debating with peers, and forming coherent arguments. The elements of this definition can be found in the described test system. The table shows the presence of the corresponding element of the definition in the three phases of the test: preparation (before), conducting (during) and verification (after).

Tab. 1: Elements of inquiry-based education in the test

Inquiry	Test	Before	During	After
diagnosing problems	understanding the item	+	+	+
critiquing experiments	using computer	+	+	-
distinguishing alternatives	comparing options	-	+	+
planning investigations	how to search	+	+	-
researching conjectures	yes, no, nothing	+	+	-
searching for information	searching	+	+	-
constructing models	programming	+	-	-
debating with peers	discussing	+	-	+
forming coherent arguments	understanding	-	-	+

The second column of the table gives the meaning of the elements of the definition in our case. A plus sign in the other columns means that this feature significantly present at the corresponding stage of the test.

3. SOFTWARE

The developed software allows the teacher easily to prepare a test bank, to generate the required number of individual tests, to upload the students' responses and to make analysis of the results for each item of the test bank.

3.1. Test generator

Preparation of the test begins with selecting items – stems and options – and put them into an item bank. At least 10 items each having at least 5-6 possible answers should be completed and stored as a text file in a particular format. This file is the input to test generator. The test generator generates individual tests using random distribution of both items and their options. Each individual test consists of 10-20 items with 4 options (a, b, c, d). The output plain text file (out.tex) contains all individual tests. The second output file (tab.tex) is a table for checking the individual tests. The third file (data.tex) is a copy of the input file with additional data for the generated individual tests. All output files are in LATEX format and have fixed as the above names.

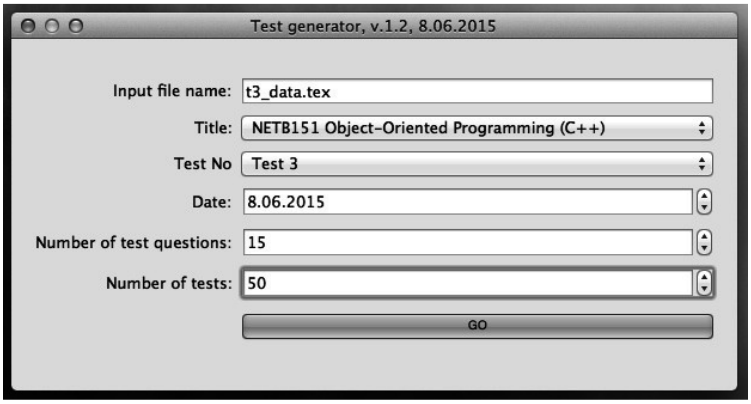

Fig. 1: The user interface of test generator.

3.2. Test checker

Checking the answers of the individual tests can be carried out manually – using the table, generated by test generator, or automatically – by test checker. Uploading data of the students' responses can be done manually – using the user interface of test checker or automatically – using a special template (on paper) for the students' responses and scanner.

The program creates a text file (save.txt), containing verified tests and can upload this file.

After entering the students' responses, test checker gives the results – for each option of each item in the test bank calculates two sets of numbers. The set $X = \{x, x_1, x_2, x_3\}$ represents all the individual tests and the set $Y = \{y, y_1, y_2, y_3\}$ represents the individual tests of students, which pass the test ($e > 2$).

- x, y – the number of individual tests which contain the corresponding item and four of its options;
- x_1, y_1 – the number of tests without response;
- x_2, y_2 – the number of tests with correct response;
- x_3, y_3 – the number of tests with incorrect (opposite) response.

The names of the output files of test checker are data_result.txt and data_result1.txt. The format of the second file is suitable for input in spreadsheet.

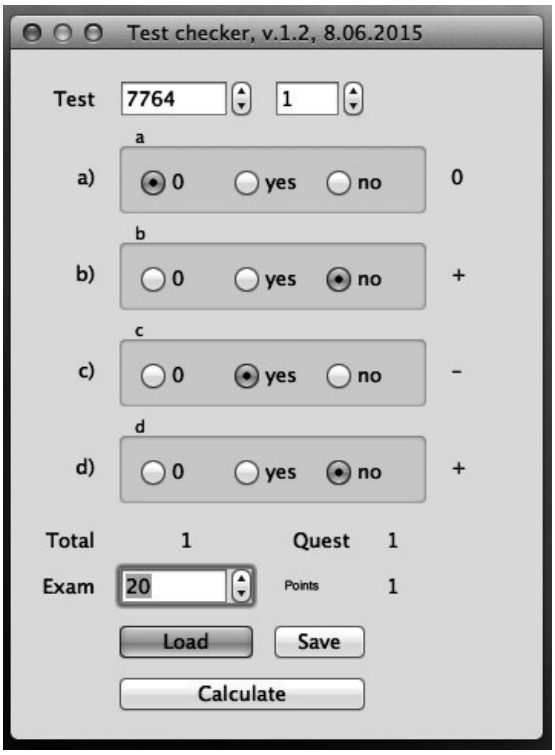

Fig. 1: The user interface of test checker.

4. CONCLUSIONS

The software is written in C++ using Qt – cross-platform application and UI development framework [5]. It is publicly available and open source (see [6] and [7]). The software tools save a lot of time and efforts of the teacher for the preparation and verification the test and evaluation the test results.

Only presented test system is not sufficient for assessing the students' knowledge of programming. However, it is an essential element of a comprehensive evaluation system that includes examination of the ability to write programs as homework and classwork [1].

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Table of Contents

Volume 1 – Mathematics and Informatics

MINI-CONFERENCE "INQUIRY-BASED APPROACH IN HIGHER EDUCATION IN MATHEMATICS AND INFORMATICS"

Table of Contents

Volume 2 – Physics and Technology

Table of Contents

Volume 3 – Geography, Ecology and Environment Protection